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BOOLEAN MODELS AND INFINITARY FIRST ORDER LANGUAGES

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Abstract. The paper develops a systematic use of boolean models in the model theory of infinitary languages. This yields a notion of (boolean) saturated model for denumerable sublanguages of $L_{\omega_1 \omega}$. The methods of saturated models are then applied; in particular results of "upward Löwenheim-Skolem" type and results relating syntactic and semantic properties are obtained.

The paper is centered on "characterization results" and on their methods of proof; we give this name to all equivalence results between a syntactic and a semantic property of formulas: in this sense, the first such result is the completeness theorem for $L_{\omega\omega}$; other examples are the completeness theorem for formulas with the quantifier Qx ("there exists uncountably many x ...'), the characterization of the formulas preserved under homorphisms by means of the positive formulas, etc ...

For (ω_1, ω)-formulas – which are the formulas considered here – one can say that two general methods have been developed: the purely syntactic method – see Feferman, [1] and the method based on "consistency properties" – see Makkai [2], Keisler [3] – and "model theoretic forcing" – see Stern, [11].

The two methods are closely related – and even dual, see Nebres, [4, p. 17]. This contrasts with the case of finite formulas, where one can also mention the methods using products of models, and saturated models; so it is natural to ask, as in Feferman, [1, p. 10], whether these other methods can be extended to the infinitary case.

The paper studies what can be done in this area by means of a systematic use of boolean models. Naturally, the first developments (§ II) are very close to "consistency properties" and "forcing", and therefore we can move quickly over these topics; but this boolean model approach also leads to a generalization for infinite formulas of the saturated models method, and this constitutes the main part of the paper. The generalization replaces saturated models by models called here Σ-saturated, which have related properties (in particular Prop. III. 2, (b) and (c) and III. 3) if one replaces $L_{\omega\omega}$ by a denumerable admissible language $L_{\mathcal{A}}$. The Σ-saturated models are boolean-valued, a necessary feature of the method except in the finitary case: because two-valued models with enough properties exist only in very particular cases.

Thus the various uses of saturated models developed in Morley-Vaught [5], Keisler [6], (and Keisler [7], see Remark V.13) receive an extension to the infinitary case. Keisler [6] obtains characterization results, by proving for each a related property of saturated models; e.g., he deduces the characterization of formulas preserved under homomorphisms from the result:

If A, B are saturated of the same power, then B is a homomorphic image of A if and only if every positive sentence true in A is true in B.

The same procedure applied with Σ-saturated models yields a method, of the same generality, for proving characterization results in $L_{\mathcal{A}}$ – see § V.

In [5] Vaught introduced the method of elementary chains of saturated (or homogeneous) models to prove his "two cardinal theorem". Adapting it to Σ-saturated models, we obtain some "upward Löwenheim-Skolem" results for $L_{\mathcal{A}}$ – telling under which conditions we can obtain from a denumerable model of a formula a non denumerable one – and deduce from them completeness results. In particular, Proposition IV.8, (a) $\Leftrightarrow$ (b), extends to admissible languages Vaught's two cardinal theorem, and especially its "upward" part, which did not follow from previously known results, such as Keisler's two cardinal theorem, [8]; Proposition IV.7.b is an axiomatization for $L_{\mathcal{A}}$ of the validity in the class of non denumerable models.

The method also applies to $L_{\omega_1\omega}(Q)$. But for this language, the main results have been obtained by the "omitting types" method (Keisler [8]) in a stronger form than the use of Σ-saturated models is able to do.

Still there is a possibility that these models become valuable in that domain: Keisler's methods do not seem to apply if one restricts oneself to fragments of $L_{\omega_1\omega}(Q)$ which are not closed under Q-quantification, and for such fragments we give a completeness result (Proposition IV.12). More precisely, we shall give an axiomatization $\mathbf{C}$ of the valid formulas of $L_{\mathscr{A}}(Q)$, which has a "*subformula property*":

For $\psi \in L_{\mathscr{A}}(Q)$, ψ *is consequence of* $\mathbf{C}$ *if and only if it is already consequence of* $\mathbf{C} \cap \{\theta \in L_{\mathscr{A}}(Q)$; *every subformula of* θ *in the form* $Qx\varphi$ *is also subformula of* $\psi\}$.

Another application concerns preservation of formulas under unions of chains of models: we obtain a simple necessary and sufficient (semantic) condition – Proposition IV.14 – and a related characterization result – Proposition V.7.

This paper is based on a part of the author's doctoral dissertation, which he prepared at the Université de Paris 7. Another part of the dissertation will be developed in a forthcoming paper which shall be a sequel to the present one. The author wishes to express his warm thanks to Professor Georg Kreisel for his constant encouragement and advice during the preparation of the thesis. Thanks are also due to Professor S. Feferman, for several remarks and for pointing out a mistake in the proof of Theorem V.12.

§ I. Preliminaries

Boolean algebras. $\mathbb{B}$, $\mathbb{B}_1$ ... denote boolean algebras, for which we use the notations $1, 0, \mathsf{C}, \cap, \cup, <$. $\mathbb{2}$ denotes the two-element algebra. For any $\mathbb{B}$, $\overline{\mathbb{B}}$ is the normal completion of $\mathbb{B}$.

If D is a (proper) filter in $\mathbb{B}$, for all $X \in \mathbb{B}$, X/D is the equivalence class modulo D of X, $\mathbb{B}/D$ is the boolean algebra of these equivalence classes, and $1/D$ denotes the homomorphism: $X \to X/D$, from $\mathbb{B}$ onto $\mathbb{B}/D$.

Formulas. We suppose the basic syntactic notions to be known, as well as the standard notations that we use for them. In addition, the following less standard notations will be used.

If L is a first order language, we denote by L_κ, L_∞ the collections of formulas usually denoted by $\mathrm{L}_{\kappa\omega}$, $\mathrm{L}_{\infty\omega}$ (because we shall not consider formulas of $\mathrm{L}_{\infty\lambda}$ for $\lambda > \omega$). $\psi(v_0 \ldots v_n)$ always denotes a formula whose free variables are among $v_0 \ldots v_n$ (n always an integer); moreover, in a given context, ψ and $\psi(v_0 \ldots v_n)$ denote the same object. If $t_0 \ldots t_n$ are terms, $\psi(t_0 \ldots t_n)$ is the formula obtained by substituting t_i for v_i in ψ. All this holds for any notation of formula *or set of formulas* in place of ψ, and of terms and variables in place of t_i, v_i. We denote by $\bigwedge$ and $\bigvee$ the quantifiers, and take $\wedge, \bigwedge\!\!\!\bigwedge, \neg$ as primitive connectives.

For brevity, we often write $\bar{x}$ for $x_0 \ldots x_{n-1}$, $\bar{x} \in X$ for $x_0 \in X \ldots x_{n-1} \in X$, $\bigvee \bar{x}$ for $\bigvee x_0 \ldots \bigvee x_{n-1}$, and so on. Moreover, when we let $\bar{x}$ vary over some set it must be understood that we let n vary over ω at the same time, except if n is explicitly kept fixed by the context. The same convention holds for any letter in place of x.

If Γ is included in L_∞, X is a set or a model, $\Gamma(X)$ denotes $\{\psi(\bar{x}\, v_n \ldots v_p) \colon \psi(v_0 \ldots v_p) \in \Gamma, \bar{x} \in X\}$. For any function $f\colon X \to Y$, for any $\psi \in \Gamma(X)$, $f\psi$ is the formula obtained by substituting $f(x)$ for x in ψ; and for any set Ψ of formulas, $f\Psi$ is $\{f\psi\colon \psi \in \Psi\}$.

For simplicity, we shall restrict us to languages L without function symbol.

Models. A $\mathbb{B}$-*valued model* M is a set, denoted by $|M|$ and called the domain of M, together with a function wich assigns to every constant c of L an element of M denoted c_M (if c itself is element of M, $c_M = c$), and with an "*interpretation of atomic sentences*", which is a function associating to every atomic sentence θ of $L_\infty(M)$ an element of $\mathbb{B}$ denoted by θ_M; this function must satisfy $\theta(c)_M = \theta(c_M)_M$, for any atomic formula $\theta(v)$ and any constant c of L.

Then for any sentence $\psi \in L_\infty(M)$, its *interpretation in M* is an element ψ_M of $\overline{\mathbb{B}}$, defined by induction on ψ:

if $\psi = \neg\varphi$, $\psi_M = \mathsf{C}\varphi_M$; if $\psi = \bigwedge\!\!\!\bigwedge \Phi$ then $\psi_M = \bigcap\{\varphi_M ; \varphi \in \Phi\}$; and if $\psi = \bigwedge v\varphi(v)$ then $\psi_M = \bigcap\{\varphi(a)_M ; a \in M\}$.

Finally, if $\psi(\bar{v}) \in L_\infty(M)$ its interpretation in M, ψ_M, is the function which to $\bar{a} \in M$ assigns $\psi(\bar{a})_M \in \overline{\mathbb{B}}$. We shall write $M \models \Phi$ or *M satisfies* Φ to mean that Φ is a sentence or a set of sentences, whose interpretation in M is the element 1 of $\mathbb{B}$.

Actually we always consider languages L containing the equality symbol, and by a model M of L it is understood *a model satisfying the equality axioms of* L. But we do not require the stronger condition: M is an *equality model,* that is:

$M \models (a = a')$ if and only if a and a' are the same element of $|M|$; this requirement is unnecessary because there is a trivial and standard procedure to transform any model of the equality axioms into an equality model, and we assume that the reader is familiar with this procedure.

We denote by $M \upharpoonright L'$ and $M \upharpoonright X$ the restrictions of a model M of L to a sublanguage L′ and to a set $X \subset |M|$.

We consider a set $\Gamma \subset L_\infty$, *closed under subformulas*: $\bigwedge\!\!\!\bigwedge \Phi \in \Gamma$ implies $\Phi \subset \Gamma$, $\neg\varphi \in \Gamma$ implies $\varphi \in \Gamma$, $\bigwedge v\varphi \in \Gamma$ implies $\varphi \in \Gamma$, and moreover Γ is closed by substitution of terms.

We say that *N is a Γ-submodel* of a model M if N is a submodel of M and if for any $\psi(v) \in \Gamma(N)$ there exists $a \in N$ such that $\bigvee v\,\psi(v)_M = \psi(a)_M$ *M is a Γ-model* if it is a Γ-submodel of itself.

Löwenheim–Skolem Theorem 1. *If N is a Γ-submodel of M then for any sentence $\psi \in \Gamma(N)$, $\psi_M = \psi_N$. If M is a Γ-model and $X \subset |M|$, there exists a Γ-submodel of M containing X of power $\leqslant |X| + |\Gamma| + \omega$.*

Admissible sets and languages. Throughout $\mathcal{L}$ shall be the language of set theory. We consider only finite formulas of $\mathcal{L}$ and for them we use the symbols $\forall$ and $\exists$ instead of $\bigwedge$ and $\bigvee$. We use also restricted quantifiers $(\forall x \in y)$ and $(\exists x \in y)$ (x, y any variables). Δ_0 denotes the set of formulas of $\mathcal{L}$ which have only restricted quantifiers, Σ the set of formulas built from Δ_0 formulas using only restricted quantifiers and existential (unrestricted) quantifiers.

We shall consider two axioms schemes, which are the universal closures of the following formulas:

Δ_0-comprehension: $\forall x\ \exists y\ \forall z\ [z \in y \leftrightarrow z \in x \wedge \theta(z\bar{u})]$ (for each Δ_0 formula θ).

Σ-reflexion: $\theta(u) \to \exists y\{(u \in y) \wedge [(\forall v_1 \in y)(\forall v_2 \in v_1)\, v_2 \in y] \wedge \theta^{(y)}(u)\}$ where θ is any Σ formula, and $\theta^{(y)}$ denotes the formula obtained by substituting everywhere in θ $(\forall v \in y)$ for $\forall v$ and $(\exists v \in y)$ for $\exists v$.

Throughout $\mathscr{A}$ denotes an admissible set: that is $\mathscr{A}$ is a transitive set, for any set $x \in \mathscr{A}$, the transitive closure of x is in $\mathscr{A}$, and $\mathscr{A}$ satisfies the Δ_0-comprehension and Σ-reflexion schemes. We assume known the basic theory of admissible sets – e.g. that Lemma 2 and that transfinite induction for properties expressed by Σ formulas are valid in $\mathscr{A}$ – see [9] for an exposition of the subject.

$X \subset \mathscr{A}^n$ is Σ if there is a Σ formula $\theta(\bar{x}\,u)$ and $a \in \mathscr{A}$ such that X is the interpretation of $\theta(\bar{x}\,a)$ in $\mathscr{A}$. X is Δ if both X and $\mathscr{A}^n - X$ are Σ.

Lemma 2. *Let $F(xy\bar{u})$ be a Σ formula; the universal closure of the following formula is true in any admissible set:*

$$(\forall x \in u_0)\, \exists y\ F(xy\bar{u}) \to \exists v[(\forall x \in u_0)(\exists y \in v)F \wedge (\forall y \in v)(\exists x \in u_0)F]\ .$$

We say that we *relativize the above notions to a predicate* G if we replace in their definitions the language $\mathcal{L}$ by the language $\mathcal{L}^G$ which contains in addition the predicate G. We use G in superscript to denote a notion relativized to G: Δ_0^G, Δ^G, Σ^G etc ...; except that admissible^G is replaced by *admissible in* G. And similarly if we relativize to several predicates $G_1, G_2, \ldots$

If L is a first order language, $L_{\mathscr{A}}$ denotes the intersection of $\mathscr{A}$ with L_∞. Whenever we consider $L_{\mathscr{A}}$, we assume that

(*) the set of relations of L, and their number of arguments, and the sets of constants and variables of L are Δ classes in $\mathscr{A}$; we also assume that L is not "too large", that is we can add infinitely many new relations and constants to L, in such a way that the extended language still satisfies (*). We say that L *is a language of* $\mathcal{A}$ if it has these properties.

For a proof of the following results, see Barwise [9].

Proposition 3. *The syntax of* $\mathrm{L}_{\mathcal{A}}$ *is* Δ *over* $\mathcal{A}$.

By this we mean e.g. that the following relations are Δ: "$\varphi \in \mathrm{L}_{\mathcal{A}}$", "$\varphi$ is a positive sentence of $\mathrm{L}_{\mathcal{A}}$", "$\varphi = \varphi(v)$, t is a term of L and $\psi = \varphi(t)$" etc..

Proposition 4 (Completeness theorem). *For sentences or sets of sentences of* L_∞, Ψ, Φ, *we write* $\Psi \vdash \Phi$ *if every* $\mathbf{2}$*-valued model which satisfies* Ψ *also satisfies* Φ. *Then if* $\mathcal{A}$ *is denumerable and* L *a language of* $\mathcal{A}$, *the relation* $\psi \vdash \varphi$ $(\psi, \varphi \in \mathrm{L}_{\mathcal{A}})$ *is* Σ.

For $\Gamma_0 \subset \mathrm{L}_\infty$, a *theory* in Γ_0 is a set T of sentences of Γ_0, which is satisfied in some model. We say that *the model* M *realizes the theory* T *in* Γ_0 if for every sentence ψ of Γ_0, $(M \models \psi) \Leftrightarrow (T \vdash \psi)$. We shall generally omit the mention "in Γ_0", the set Γ_0 being indicated by the context. Note that "M realizes T" is a stronger property than "M satisfies T".

§ II. Quotients, canonical models, compactness

We prove a "compactness lemma", which gives the Barwise compactness theorem when applied to admissible languages, but is valid for any denumerable *fragment* of L_{ω_1}, in the sense of [3, p. 17]. This lemma is what will be needed to show the existence of Σ-saturated models (prop. III.1) in way similar to the construction, given in [5], of saturated models. Here we give one other use of it, which allows us to extend a result of Kreisel [10] – on the definability of the elements of a structure rigidly contained in a theory – to denumerable theories in L_{ω_1}.

We also prove a characterization result relative to direct products (prop. 8).

But the reader familiar with the subject will find that this paragraph largely consists in exposing, from a non customary point of view, well known results or methods.

Quotients of boolean models. We denote by Γ_0 a fixed but arbitrary denumerable fragment of L_{ω_1}.

Taking the problem (*) as example, we first want to indicate some features of the use of boolean models:

(*) *given* $\psi_1, \psi_2 \in \Gamma_0$, *construct a model of* ψ_1, *a model of* ψ_2, *and a homomorphism between the two.*

Boolean models facilitate (*) by allowing to introduce "weak notions" of homomorphism, which are easier to construct than the true one (actually they suggest three different definitions of a "weak homomorphism" f between models M_1, M_2, which coincide with the true notion if M_1, M_2 are two-valued, but are weaker and distinct properties if M_1, M_2 are boolean:

(A) *for any positive* $\theta \in \Gamma_0(M_1), M_1 \models \theta \Rightarrow (f\theta)_{M_2} \neq 0$

(B) *for any positive* $\theta \in \Gamma_0(M_1), M_1 \models \theta \Rightarrow M_2 \models f\theta$

(C) *for any positive* $\theta \in \Gamma_0(M_1), M_1 \models \theta \Rightarrow M_2 \models f\theta$, and $M_2 \models \neg f\theta \Rightarrow M_1 \models \neg\theta$)

Thus one solves (*) in two steps: first, one constructs models M_ϵ satisfying ψ_ϵ ($\epsilon = 1, 2$), and a "weak homomorphism" f (in one of the senses (A), (B), (C)) between M_1, M_2. Second, one makes (for $\epsilon = 1, 2$) a suitable transformation, called quotient, of M_ϵ, which yields models N_ϵ of ψ_ϵ, such that f becomes a true homomorphism between them.

Many other constructions in addition to the solution of (*) will present the same features: a first step where the construction is realized up to the point that "weak relations" between models hold instead of the intended ones. A second step in which these weak relations become the true ones between suitable quotients of the models constructed in the first step. We now study these *quotients of boolean models* used in the second step. In particular we give a general lemma which takes care of the step in most cases; this lemma is divided in three parts A, B, C, each corresponding to one of the three kinds of "weak relations between models" exemplified respectively by the definitions (A), (B), (C) above.

Note that Lemmas A, B, C, are dissimilar: the choice between the three kinds leads to essentially different proofs (in the first step as well as in the second).

Definition 1. Let M be a $\mathbb{B}$-valued model of L, D a filter in $\mathbb{B}$. We denote by M/D the $\mathbb{2}$-valued model of L of domain $|M|$ which gives the same interpretation as M to constants of L, and such that for every atomic sentence ψ of $\Gamma_0(M), M/D \models \psi \Leftrightarrow \psi_M \in D$. Given $\Gamma \subset \Gamma_0$ we say that M/D *is a* Γ-*quotient of* M if $M/D \models \psi \Leftrightarrow \psi_M \in D$ for any sentence ψ of $\Gamma(M)$.

We note $M/\bar{D}$ the $\mathbb{B}/D$-valued model of L of domain $|M|$ which gives the same interpretation as M to constants of L and such that for every atomic sentence ψ of $\Gamma_0(M)$, $\psi_{M/\bar{D}} = (\psi_M)/D$.

Given $\Gamma \subset \Gamma_0$ we say that $M/\bar{D}$ *is a* Γ-*quotient of* M if $\psi_{M/\bar{D}} = \psi_M/D$ for any sentence $\psi \in \Gamma(M)$.

Lemma 2. *Let M, D, be as in Def. 1, and let* $\Psi = \{\psi \in \Gamma_0(M) : \psi_M \in D\}$.

(a) *Suppose that whenever* $\bigvee \Phi \in \Psi$, Ψ *contains a* $\varphi \in \Phi$, *and whenever* $\mathbf{V}\, v\varphi \in \Psi$, *there is an* $a \in M$ *such that* $\varphi(a) \in \Psi$. *Then* M/D *is a* Γ_0-*quotient of* M (*and conversely*).

(b) *Suppose that* Ψ *is closed under* $\bigwedge$ (*that is* $\bigwedge \Phi \in \Gamma_0$ *and* $\Phi \subset \Psi \Rightarrow \bigwedge \Phi \in \Psi$), *and that if* $\{\varphi(a) : a \in M\} \subset \Psi$, *then* $\mathbf{\Lambda}\, v\varphi(v) \in \Psi$. *Then* $M/\bar{D}$ *is a* Γ_0-*quotient of* M (*and conversely*).

Proof. The two proofs are similar, we show (b), that is $\psi_{M/\bar{D}} = \psi_M/D$, by induction on ψ:

it is true for atomic ψ, by definition of $M/\overline{D}$; we now take ψ arbitrary and assume it true for its subformulas;

if $\psi = \neg\varphi$, it follows because $1/D$ is a homomorphism;

if $\psi = \bigwedge\!\!\!\bigwedge \Phi$, assume θ is a sentence of $\Gamma_0(M)$ such that $\theta_M/D < \varphi_M/D$ – that is $(\theta \to \varphi) \in \Psi$ – for all $\varphi \in \Phi$; if we show that it implies $\theta_M/D < \bigwedge\!\!\!\bigwedge \Phi_M/D$, it will be clear that $\bigwedge\!\!\!\bigwedge \Phi_M/D = \bigcap\{\varphi_M/D; \varphi \in \Phi\}$ which is what we want; now if $(\theta \to \varphi) \in \Psi$ for all $\varphi \in \Phi$, then by the closure of Ψ under $\bigwedge\!\!\!\bigwedge$, $\bigwedge\!\!\!\bigwedge\{\theta \to \varphi; \varphi \in \Phi\} \in \Psi$, so $\theta \to \bigwedge\!\!\!\bigwedge \Phi \in \Psi$, that is $\theta_M/D < (\bigwedge\!\!\!\bigwedge\Phi)_M/D$. If $\psi = \bigwedge v\varphi(v)$, the proof is similar to the preceding case (as if $\bigwedge v\varphi(v)$ were $\bigwedge\!\!\!\bigwedge\{\varphi(a); a \in M\}$).

Lemma A (Consistency properties) [1]. *Let M_ϵ $(\epsilon \in \{0 \dots n-1\})$ be denumerable models of* L *and E a set whose elements are sequences $\overline{s} = (s_\epsilon : \epsilon < n)$ such that s_ϵ is a sentence of $\Gamma_0(M_\epsilon)$ and $(s_\epsilon)_{M_\epsilon} \neq 0$. Assume, for every $\overline{s} \in E$, and every $\epsilon < n$:*

(i) *if $s'_\epsilon \in \Gamma_0(M_\epsilon)$ and $M_\epsilon \models s_\epsilon \to s'_\epsilon$, then $(s_0, \dots, s'_\epsilon, \dots, s_n) \in E$*

(ii) *if $\bigvee\!\!\!\bigvee_I \psi^i$ is a valid sentence of $\Gamma_0(M_\epsilon)$, then there is $i_0 \in I$ such that $(s_0, \dots, s_\epsilon \wedge \psi^{i_0}, \dots, s_{n-1}) \in E$*

(iii) *if $\bigvee v\, \psi(v)$ is a valid sentence of $\Gamma_0(M_\epsilon)$, then there is $a \in M_\epsilon$ such that $(s_0, \dots, s_\epsilon \wedge \psi(a), \dots, s_{n-1}) \in E$.*

Then there exists ultrafilters D_ϵ $(\epsilon < n)$ such that M_ϵ/D_ϵ is a Γ_0-quotient of M_ϵ and for any sequence of sentences $\overline{s} \in \prod_{\epsilon<n} \Gamma_0(M_\epsilon)$ $\bigwedge\!\!\!\bigwedge_{\epsilon<n} (M_\epsilon/D_\epsilon \models s_\epsilon)$ implies $\overline{s} \in E$.

Proof. One defines by induction a sequence $\{R^k; k < \omega\} \subset E$: assume $R^0, \dots, R^{k-1}$ chosen, and that $R^{k-1} = (s_0, \dots, s_{n-1})$. If k is even one fixes $\epsilon < n$ and a valid sentence $\bigvee\!\!\!\bigvee_I \psi^i$ of $\Gamma_0(M_\epsilon)$; by (ii) there exists $i \in I$ such that $(s_0, \dots, s_\epsilon \wedge \psi^i, \dots, s_{n-1}) \in E$, and one takes this sequence as R^k. If k is odd, one fixes $\epsilon < n$ and a valid sentence $\bigvee v\, \psi(v)$ of $\Gamma_0(M_\epsilon)$; by (iii) there exists $a \in M_\epsilon$ such that $(s_0, \dots, s_\epsilon \wedge \psi(a), \dots, s_{n-1}) \in E$, and one takes this sequence as R^k.

Let D_ϵ $(\epsilon < n)$ be the filter generated by $\{(s^k_\epsilon)_{M_\epsilon}; k \in \omega\}$, where $R^k = (s^k_0, \dots, s^k_{n-1})$. One easily sees that the degrees of freedom in the choice of $\{R^k; k < \omega\}$ can be used to the effect that D_ϵ satisfies the hypothesis of Lemma 2.a: for it is enough to take care that the valid

[1] This lemma is a reformulation of the Main Lemma in Makkai [2].

sentences of the form $\bigvee_i \psi^i$ which we chose for every even k include all such formulas of $\Gamma_0(M_\epsilon)$, and the valid sentences $\mathbf{V} v \psi(v)$ which we chose for every odd value of k include all such formulas of $\Gamma_0(M_\epsilon)$. Then by Lemma 2, M_ϵ/D_ϵ are Γ_0-quotients of M_ϵ, and using the property (i) of E it is clear that they have the required property.

Applying Lemma A to a single model M, with $E = \{\theta \in \Gamma_0(M)\colon \theta_M \neq 0\}$, gives as a corollary:

Rasiowa–Sikorski Theorem 3. *For any denumerable $\mathbb{B}$-model M, there exists an ultrafilter D in $\mathbb{B}$ such that M/D is a Γ_0-quotient of M.*

We shall often use this theorem implicitly: having constructed denumerable boolean models with certain properties, we shall admit that two-valued models with the same properties exist, as it will follow trivially from this theorem.

We shall always denote by $\mathbf{T}$ the true formula and by $\perp$ the false formula.

For any $\Gamma \subset \Gamma_0$, H_Γ is the set of "*Horn formulas in* Γ", precisely the set of formulas of Γ_0 that are built using only conjunction and existential quantification from the formulas of the form

$$\psi_0 \wedge \ldots \wedge \psi_n \rightarrow \varphi, \text{ where } \{\psi_0, \ldots, \psi_n, \varphi\} \subset \Gamma \cup \{\mathbf{T}, \perp\}.$$

Lemma B. (a) *Assume M is a Γ_0-model and the formula $v_0 = v_1$ is in Γ. Then* (i) $\Rightarrow$ (ii), *and the converse is true provided $M \models \mathbf{V} x \mathbf{V} y\ x \neq y$:* (i) *$M/D$ is a Γ-quotient of M;* (ii) *for all $\psi \in H_\Gamma(M)$, $M \models \psi \Rightarrow M/D \models \psi$.*

(b) *Let $\Gamma \subset \Gamma_0$ contain all atomic formulas, and f be a map from M onto a $\mathbb{2}$-valued model N, such that for all $\psi \in H_\Gamma(M), M \models \psi \Rightarrow N \models f\psi$. Then there exists D such that M/D is a Γ-quotient of M and is isomorphic to N modulo f.*

Proof. (ii) $\Rightarrow$ (i) Assuming M a Γ_0-model of $\mathbf{V} x \mathbf{V} y\ x \neq y$, for any sentence $\psi \in \Gamma(M)$ there exist $a_1, a_2 \in M$ such that M satisfies $(a_1 = a_2) \leftrightarrow \psi$; by (ii) M/D satisfies the same. So $M/D \models \psi \Leftrightarrow M/D \models a_1 = a_2 \Leftrightarrow$ $\Leftrightarrow (a_1 = a_2)_M \in D \Leftrightarrow \psi_M \in D$.

(i) $\Rightarrow$ (ii) We prove even, by induction on $\psi \in H_\Gamma(M)$, that

(ii)′ $\psi_M \in D \Rightarrow M/D \models \psi$

In the initial case where $\psi = \varphi_0 \wedge \ldots \wedge \varphi_n \to \varphi$ and $\{\varphi_0, \ldots, \varphi\} \subset \Gamma \cup \cup \{\top, \bot\}$ assume that $M/D \vDash \neg\psi$; then by (i) $(\varphi_0 \wedge \ldots \wedge \varphi_n)_M \in D$ and $\varphi_M \notin D$, hence $\psi_M \notin D$: (ii)$'$ is shown in contrapositive form in this case.

The case $\psi = \bigwedge\Phi$ of the induction is trivial. And in the case $\psi = \bigvee v\varphi(v)$, consider $a \in M$ such that $\varphi(a)_M = \bigvee v\varphi(v)_M$: $\psi_M \in D$ implies $\varphi(a)_M \in D$, hence by the induction hypothesis $M/D \vDash \varphi(a)$; so $M/D \vDash \psi$.

(b) Let ψ be a finite conjunction of sentences of $\Gamma(M)$, such that $N \vDash f\psi$; then $\psi_M \neq 0$ for otherwise, since $\neg\psi \in H_\Gamma(M)$, $N \vDash f\neg\psi$, a contradiction. So the filter D generated by $\{\psi_M\, ;\, \psi$ is such a sentence$\}$ is proper. Now let ψ be a sentence of $\Gamma(M)$ such that $\psi_M \in D$; by definition of D, there is a finite conjunction φ of sentences of $\Gamma(M)$ such that $N \vDash f\varphi$ and $M \vDash \varphi \to \psi$; then since $\varphi \to \psi$ belongs to $H_\Gamma(M)$, the assumption on f implies: $N \vDash f(\varphi \to \psi)$, hence $N \vDash f\psi$. Assuming $\psi_M \in D$ we proved $N \vDash f\psi$; since the converse is true by definition of D, we proved:

(1) for any sentence ψ of $\Gamma(M)$, $\psi_M \in D \Leftrightarrow N \vDash f\psi$.

(1) holds for atomic sentences ψ, as they belong to Γ, which implies that f is an isomorphism from M/D onto N. This implies:

(2) for any sentence $\psi \in \mathrm{L}_\infty(M)$, $M/D \vDash \psi \Leftrightarrow N \vDash f\psi$.

(1) and (2) together imply that for any sentence ψ of $\Gamma(M)$, $M/D \vDash \psi \Leftrightarrow \psi_M \in D$: M/D is a Γ-quotient of M, which ends the proof.

Let f be a partial mapping from a $\mathbb{B}$-valued model M onto a $\mathbb{B}_1$-valued model N, such that there exists an isomorphism q from $\mathbb{B}$ onto $\mathbb{B}_1$, satisfying: for every atomic sentence θ of $\Gamma_0(\mathrm{dom}\, f)$

$$(f\theta)_N = q(\theta_M) \ \text{(respectively: } (f\theta)_N < q(\theta_M)) \ ;$$

such a map is called a *relative* isomorphism, or isomorphism *relative to q* (respectively: *relative* homomorphism, or homomorphism *relative to q*). We shall of course omit the word relative if $\mathbb{B} = \mathbb{B}_1$ and $q = 1_B$; and in fact we shall omit it when the context permits replacing M by its "quotient" M/q. (We use M/q to denote the $\mathbb{B}_1$-valued model with same domain and interpretation of constants as M, such that $\theta_{M/q} = q(\theta_M)$ for all atomic sentences θ).

But note that it is not always possible to replace M by M/q: e.g., if M is to be a submodel of N, we cannot replace M by M/q as soon as $q \neq 1_{\mathbb{B}}$, for then M/q is not a submodel of N. Such cases will arise in the proof of Propositions IV.3 and IV.10, and in the statement of Proposition V.6.

Lemma C. *Let Γ be a subset of Γ_0, containing the atomic formulas and closed under boolean operations; let f be an application from M into a $\mathbb{B}_1$-valued model N, such that for any $\theta \in \Gamma(M)$, $M \models \theta \Leftrightarrow N \models f\theta$; assume also that*
(*) for any $X \in \mathbb{B}$, any $X_1 \in \mathbb{B}_1$, there exist $\theta \in \Gamma(M)$, $\theta_1 \in \Gamma(\text{im.} f)$ such that $\theta_M = X$ and $\theta_{1N} = X_1$. Then
(a) *there is a unique isomorphism q between $\mathbb{B}$ and $\mathbb{B}_1$ satisfying $q(\theta_M) = (f\theta)_N$ for any $\theta \in \Gamma(M)$; and*
(b) *f is an isomorphism from M into N, relative to q.*

Proof. The proof of (a) is similar to that of Lemma B.b. (b) is true then since $q(\theta_M) = (f\theta)_N$ for any atomic θ.

Compactness. Let T be a theory in Γ_0, and C an infinite set of constants not in L. For any sentence $\psi \in \Gamma_0(C)$, ψ/T denotes the set of sentences φ of $\Gamma_0(C)$ such that $T \vdash \psi \leftrightarrow \varphi$. If we set $\mathrm{C}(\psi/T) = (\neg\,\psi)/T$ and $(\psi/T) \cap (\varphi/T) = (\psi \wedge \varphi)/T$, we can define on $\{\psi/T;\ \psi$ a sentence of $\Gamma_0(C)\}$ a boolean algebra called the *Lindenbaum algebra of T*.

We call *canonical model of T* the model M with values in the Lindenbaum algebra of T, such that $|M|$ is the union of C and of the set of constants of L, and for any atomic sentence $\psi \in \Gamma_0(M)$, $\psi_M = \psi/T$. M is indeed a model of T by

Lemma 4. *For any sentence $\psi \in \Gamma_0(C)$, $\psi_M = \psi/T$. So M realizes the theory T.*

Proof. Straightforward induction on ψ.

Remark. Canonical models have a "universal property" which underlies their usefulness: *a $\mathbf{2}$-valued model A of power $\leq |C|$ satisfies T if and only if it is isomorphic to a Γ_0-quotient of the canonical model M of T.*

To see this let f be any surjection of M onto A, such that $f(c) = c_A$ for all constants c of L; if A satisfies T, Lemma B.b applied with $\Gamma = \Gamma_0 = H_{\Gamma_0}$ gives a Γ_0-quotient M/D such that f is an isomorphism between M/D and A.

Compactness Lemma 5. *Let T be a theory in Γ_0. We write $T \vdash^* \varphi$ for "$(\exists\theta \in T)\theta \vdash \varphi$".*
If $\{\varphi\colon T \vdash^* \varphi, \varphi$ a sentence of $\Gamma_0\}$ is closed under $\bigwedge$, then

$T \vdash \varphi \Leftrightarrow T \vdash^* \varphi$, for any $\varphi \in \Gamma_0$; in particular ($\varphi = \perp$) T has a model if and only if every formula of T has one.

Proof. We can restrict ourselves to the case that any formula of T has a model, that is $T \vdash^* \perp$ does not hold. Let M be the canonical model of the true formula, and D be the filter in its boolean algebra generated by $\{\varphi_M : T \vdash^* \varphi, \varphi$ a sentence of $\Gamma_0(C)\}$. We need only to show that $M/\bar{D}$ is a Γ_0-quotient of M: then for $\varphi \in \Gamma_0(C), M/\bar{D} \models \varphi \Leftrightarrow \varphi_M \in D \Leftrightarrow T \vdash^* \varphi$, from which the conclusion will follow.

So it is enough to show that D, M verify the hypothesis of Lemma 2.b:

Firstly the set Ψ of the Lemma, which here equals $\{\psi \in \Gamma_0(C) : T \vdash^* \psi\}$ is closed under $\bigwedge\!\!\bigwedge$; for suppose $\bigwedge\!\!\bigwedge \Phi(\bar{v}) \in \Gamma_0$, $\bar{c} \in C$, and $T \vdash^* \varphi(\bar{c})$ for all $\varphi \in \Phi$; then for all $\varphi \in \Phi$, $T \vdash^* \bigwedge \bar{v}\, \varphi(\bar{v})$; then, by hypothesis, $\bigwedge\!\!\bigwedge \{\bigwedge \bar{v}\, \varphi(\bar{v}) : \varphi \in \Phi\} \in \Gamma_0$ and $T \vdash^* \bigwedge\!\!\bigwedge \{\bigwedge \bar{v}\, \varphi(\bar{v}) : \varphi \in \Phi\}$, so $T \vdash^* \bigwedge\!\!\bigwedge \Phi(\bar{c})$.

Secondly, this set Ψ has the other required property; for if $\psi(v_0) \in \Gamma_0(C)$ and for all $c \in C$ $T \vdash^* \psi(c)$, then $T \vdash^* \psi(c_0)$ for some c_0 not occuring in $\psi(v)$, so $T \vdash^* \bigwedge v_0\; \psi(v_0)$.

Corollary 5. *Let T be a set of sentences of Γ_0, closed under $\bigwedge\!\!\bigwedge$ and maximal in the sense: $\psi \in T$ or $\neg \psi \in T$, for any sentence ψ. Then T has a model if and only if every formula of T has one.*

Proof. It is easy to verify that $\{\varphi : T \vdash^* \varphi\}$ is closed under $\bigwedge\!\!\bigwedge$ so that Lemma 5 applies.

From now on, $\mathcal{A}$ is a denumerable admissible set, $L_{\mathcal{A}}$ a language of $\mathcal{A}$, and we study $L_{\mathcal{A}}$ instead of the more general fragment Γ_0.

Barwise compactness theorem 6. *Let T be a Σ set of sentences of $L_{\mathcal{A}}$, closed under $\bigwedge\!\!\bigwedge$. T has a model if and only if every formula of T has one.*

Proof. It suffices to apply Lemma 5 with $\Gamma_0 = L_{\mathcal{A}}$, and, to that end, to show that $\{\varphi \in L_{\mathcal{A}} : T \vdash^* \varphi\}$ is closed under $\bigwedge\!\!\bigwedge$.

There exists a Σ formula $F(\theta, \varphi)$ expressing inside $\mathcal{A}$ that $\theta, \varphi \in L_{\mathcal{A}}$, $\theta \in T$ and $\theta \vdash \varphi$. Let $\bigwedge\!\!\bigwedge \Phi$ be a sentence of $L_{\mathcal{A}}$; the relation: $\Phi \subset \{\varphi : T \vdash^* \varphi\}$ is expressed by $(\forall \varphi \in \Phi)\, \exists \theta\, F(\theta, \varphi)$. By Lemma I.2, there exists $\Theta \in \mathcal{A}$ such that $(\forall \theta \in \Theta)(\exists \varphi \in \Phi) F(\theta, \varphi)$ and $(\forall \varphi \in \Phi)$ $(\exists \theta \in \Theta) F(\theta, \varphi)$. So $\Theta \subset T$ and $\bigwedge\!\!\bigwedge \Theta \vdash \bigwedge\!\!\bigwedge \Phi$; then $(\bigwedge\!\!\bigwedge \theta) \in T$, hence $(\bigwedge\!\!\bigwedge \Phi) \in \{\varphi : T \vdash^* \varphi\}$.

Rigidly contained structures. Let T_0 be a denumerable theory in L_{ω_1}. A $\mathbb{2}$-valued model A is *rigidly contained in* T_0 if for any $\mathbb{2}$-valued model B of T_0 there exists a unique isomorphism from A into B. We denote by Γ_1 the set of formulas of L_{ω_1} built only with $\bigvee$ and $\bigwedge$ from the atomic formulas and their negations.

Proposition 7. *If A is rigidly contained in T_0, then for any $a_0 \in A$ there is $\theta_{a_0}(v_0) \in \Gamma_1$ such that $A \models \theta_{a_0}(a_0)$ and $T_0 \vdash \bigvee! \, v_0 \; \theta_{a_0}(v_0)$* [2].

Proof. We let F vary over the finite subsets of $|A|$. There exists a denumerable set $\mathcal{A}$ such that $T_0 \subset L_{\mathcal{A}}$, and $\mathcal{A}$ is admissible in T_0 and S_F for every F, where S_F denotes the intersection of $\mathcal{A}$ with $\{\theta \in \Gamma_1(F) \colon A \models \theta\}$. Let Γ be a fragment of $L_{\mathcal{A}}$ such that $T_0 \subset \Gamma$; let $f \colon A \to A'$ be an isomorphism of A onto a model whose domain is disjoint from $|A|$.

Claim. *The hypothesis of Lemma 5 holds when Γ_0 is $\Gamma(|A| \cup |A'|)$ and T is $\{\varphi \in \Gamma(|A| \cup |A'|)$: there exist F and $\theta \in S_F$ such that T_0, $\theta \wedge f\theta \vdash \varphi\}$.*

Proof. T is equal to $\{\varphi \in \Gamma(|A| \cup |A'|); T \vdash^* \varphi\}$, so that we have to consider an arbitrary subset Φ of T whose conjunction is a formula of $\Gamma(|A| \cup |A'|)$, and to prove that $\bigwedge \Phi \in T$.

Given such a set Φ, there exists an F such that $\bigwedge \Phi \in \Gamma(F \cup f(F))$. Then if $\varphi \in \Phi$, there exists $\theta \in S_F$ such that T_0, $\theta \wedge f\theta \vdash \varphi$: firstly $\varphi \in T$, hence there exists F' and $\theta' \in S_{F'}$ such that T_0, $\theta' \wedge f\theta' \vdash \varphi$; secondly we can write θ' as $\theta''(\bar{x})$ where $\theta''(\bar{v}) \in \Gamma(F \cup f(F))$ and $\bar{x} \in F' - F$, and then $\theta = \bigvee \bar{v}\, \theta''(\bar{v})$ is an element of S_F such that T_0, $\theta \wedge f\theta \vdash \varphi$.

So if $G(\theta, \varphi)$ is a Σ^{T_0, S_F} formula expressing inside $(\mathcal{A}, T_0, S_F)$ that $\theta \in S_F$ and T_0, $\theta \wedge f\theta \vdash \varphi$, the assumption $\Phi \subset T$ implies:

$$\forall \varphi \in \Phi \; \exists \theta \; G(\theta, \varphi) \,.$$

[2] It would be interesting to have a bound on the complexity of θ_{a_0}; the proof of this proposition gives a bound, but one which depends on the "complexity of A", not only on T_0. However using this bound one gets the following corollary: *if T_0 is hereditarily denumerable in the universe of constructible sets, and has a hard core* (= largest structure r.c. in T_0), *then the hard core is constructible.* This was also proved by F. Ville.

By Lemma I.2 there exists $\Theta \in \mathscr{A}$ such that

$$(\forall \varphi \in \Phi)(\exists \theta \in \Theta)\, G(\theta, \varphi) \quad \text{and} \quad (\forall \theta \in \Theta)(\exists \varphi \in \Phi)\, G(\theta, \varphi)\ ;$$

hence T_0, $\bigwedge\Theta \wedge f \bigwedge\Theta \vdash \bigwedge\Phi$ and $\bigwedge\Theta \in S_F$, so $\bigwedge\Phi \in T$.

Let $a_0 \in A$; T contains T_0 and the diagrams of A and A', so if A is rigidly contained in T_0, $T \vdash a_0 = f(a_0)$. By Lemma 5, $T \vdash^* a_0 = f(a_0)$, that is: there exists F, $\theta(\bar{a}) \in S_F$ such that $T_0 \vdash \theta(\bar{a}) \wedge f\theta(\bar{a}) \to (a_0 = = f(a_0))$; then (assuming $\theta(\bar{v}) \in \Gamma_1$) if we set $\theta_{a_0}(v_0) = \mathbf{V} v_1 \ldots \mathbf{V} v_{n-1}\, \theta(v_0 \ldots v_{n-1})$, the conclusion of the proposition holds.

Using canonical models. We use these models together with Lemma A to prove a characterization result (Proposition 8). This method is essentially a reformulation into the boolean framework of Makkai's "Consistency Properties" method, [2]. Moreover, another reformulation in terms of "forcing" was given by Stern, [11], to which the present one is equivalent, as far as concerns characterization results. So we shall restrict our exposition of the method to single application of Proposition 8.

Following Weinstein, we write $\psi \times \varphi \Rightarrow \theta$ – ψ, φ, θ being sentences of $L_{\mathscr{A}}$ – to mean: "*θ holds in any direct product of a model of ψ and a model of φ*". Thus $\psi \times \psi \Rightarrow \psi$ means that *ψ is preserved under direct products.* We define a ternary relation Γ between formulas of $L_{\mathscr{A}}$ with the same free variables, by the following inductive clauses:
– for any atomic formula θ, (θ, θ, θ), $(\neg\theta, \theta, \neg\theta)$, $(\theta, \neg\theta, \neg\theta)$ and $(\neg\theta, \neg\theta, \neg\theta)$ are in Γ
– if $\{(\psi_i, \varphi_i, \theta_i)\colon i \in I\} \subset \Gamma$, then $(\bigwedge_I \psi_i, \bigwedge_I \varphi_i, \bigwedge_I \theta_i)$, $(\bigwedge_I \psi_i, \bigvee_I \varphi_i, \bigvee_I \theta_i)$ and $(\bigvee_I \psi_i, \bigwedge_I \varphi_i, \bigvee_I \theta_i)$ are in Γ
– if $(\psi, \varphi, \theta) \in \Gamma$ and v is a variable, $(\mathbf{V} v\psi, \mathbf{V} v\varphi, \mathbf{V} v\theta)$ and $(\mathbf{\Lambda} v\psi, \mathbf{\Lambda} v\varphi, \mathbf{\Lambda} v\theta)$ are in Γ.

Proposition 8. *Following are equivalent*: (a) $\psi \times \varphi \Rightarrow \theta$ (b) *there exists* $(\psi', \varphi', \theta') \in \Gamma$ *such that* $\psi \vdash \psi'$, $\varphi \vdash \varphi'$ *and* $\theta' \vdash \theta$ [3].

[3] When ψ, φ, θ belong to $L_{\omega\omega}$, Weinstein [16] obtained a characterization of the relation $\psi \times \varphi \Rightarrow \theta$ in a much stronger sense: by giving a relation Γ such that $\psi \times \varphi \Rightarrow \theta$ if and only if there exists $(\psi', \varphi', \theta')$ in Γ such that ψ, φ, θ are equivalent respectively to ψ', φ', θ' (the relation Γ, though primitive recursive, is not given "syntactically"). The proof of Weinstein's result is based on the theorem of Feferman and Vaught, which does not generalize to L_{ω_1}, by a counterexample of Malitz, [14].

Proof. (b) $\Rightarrow$ (a). By induction on the clauses defining Γ, one proves that $(\psi', \varphi', \theta') \in \Gamma$ implies $\psi' \times \varphi' \Rightarrow \theta'$, hence (b) $\Rightarrow$ (a).

$\neg$(b) $\Rightarrow$ $\neg$(a). Assuming $\neg$(b), we take denumerable canonical models M_0 realizing the theory $\{\psi\}$, M_1 realizing $\{\varphi\}$, and M_2 realizing $\{\neg\theta\}$. We let $\{a_n ; n < \omega\}$, $\{b_n ; n < \omega\}$, $\{c_n ; n < \omega\}$ enumerate (with repetitions) their domains, in such a way that the application f: $(a_n, b_n) \to c_n$ $(n < \omega)$ is a bijection between $|M_0| \times |M_1|$ and $|M_2|$, and $f(c_{M_0}, c_{M_1}) = c_{M_2}$ for every constant c of L. And we set $E = \{(s_0, s_1, s_2)$: $s_\epsilon \in \mathrm{L}_{\mathscr{A}}(M_\epsilon)$, $(s_\epsilon)_{M_\epsilon} \neq 0$, and there does not exist $(\psi'(\bar{v}), \varphi'(\bar{v}), \theta'(\bar{v})) \in \Gamma$ such that $M_0 \models s_0 \to \psi'(\bar{a})$, $M_1 \models s_1 \to \varphi'(\bar{b})$ and $M_2 \models s_2 \to \neg\theta'(\bar{c})\}$.

Claim. *E satisfies the hypothesis of Lemma A (when $\Gamma_0 = \mathrm{L}_{\mathscr{A}}$).*

We leave to the reader the checking of the claim. Applying Lemma A as allowed by the claim, we obtain quotients M_ϵ/D_ϵ of M_ϵ $(\epsilon < 2)$. Then $M_0/D_0 \models \psi$, $M_1/D_1 \models \varphi$, $M_2/D_2 \models \neg\theta$.

Moreover, let $F(\bar{v})$ be an atomic formula of L; if $M_0/D_0 \models F(\bar{a})$ and $M_1/D_1 \models F(\bar{b})$, then $M_2/D_2 \models F(\bar{c})$, because otherwise, by the choice of D_0, D_1, D_2, $(F(\bar{a}), F(\bar{b}), \neg F(\bar{c}))$ would be an element of E, which contradicts the definition of E; in a similar way, one proves the converse: if $M_2/D_2 \models F(\bar{c})$ then $M_0/D_0 \models F(\bar{a})$ and $M_1/D_1 \models F(\bar{b})$; this shows that f is an isomorphism from $(M_0/D_0) \times (M_1/D_1)$ onto M_2/D_2; $\neg$(b) $\Rightarrow$ $\neg$(a) is thus proved.

§ III. Σ-saturated models

We define the "Σ-saturated models" of $L_{\mathcal{A}}$, and prove their existence (Proposition 1) and uniqueness (Proposition 3), and their other main properties (Proposition 2).

Definitions. Consider a model M of L, and $X \subset |M|$. $p(\bar{v})$ *is a Σ-type* (*in M*) if there exists a finite set $F \subset |M|$ such that:

$p(\bar{v})$ is a Σ subset of $L_{\mathcal{A}}(F)$, closed under $\bigwedge$, and $M \models \bigvee \bar{v}\, \theta(\bar{v})$, for all $\theta \in p$.

A sequence $\bar{a} \in M$ *realizes p over X* (*in M*) if for all $\psi(\bar{v}) \in L_{\mathcal{A}}(X)$, $M \models \psi(\bar{a}) \Leftrightarrow$ there is $\theta \in p$ such that $M \models \bigwedge \bar{v}(\theta \to \psi)$ (Note that "$\bar{a}$ realizes p" is stronger than "$\bar{a}$ satisfies p", in the same way as "M realizes T" is stronger than "M satisfies T").

M is *Σ-saturated* if

(1.i) *there is an enumeration* $\{a_\alpha ; \alpha < \gamma\}$ *of* $|M|$ *such that for all* $\alpha < \gamma$, a_α *realizes some Σ type over* $\{a_\beta ; \beta < \alpha\}$;

(1.ii) *for every set* $X \subset |M|$ *of power* $< \|M\|$, *and every Σ type* $p(v_0)$ *there is* $a \in M$ *realizing p over X.*

Proposition 1. *Let T be a Σ theory in* $L_{\mathcal{A}}$. *In every regular cardinal* λ *there exists a Σ-saturated model realizing the theory T.*

Proof. Let $\{a_\alpha ; \alpha < \lambda\}$ be a set of constants not in L, of regular cardinal λ. We define inductively sets of sentences $T_\alpha \subset L_{\mathcal{A}}(\{a_\beta ; \beta < \alpha\})$, for any $\alpha \leq \lambda$:

T_0 is the set of consequences of T in $L_{\mathcal{A}}$; if α is a limit, $T_\alpha = \bigcup_{\beta < \alpha} T_\beta$;

assume T_α defined; to define $T_{\alpha+1}$, choose a set, denoted by Θ_α, among all sets $\Theta(v)$ satisfying the condition (*) below, with $\beta_n < \alpha$

(*) there exists $\beta_1 < ... < \beta_n$ such that $\Theta(v)$ is a Σ subset of $L_{\mathcal{A}}(\{a_{\beta_1} ... a_{\beta_n}\})$, closed under $\bigwedge$; and for all $\theta \in \Theta$, $\bigvee v\theta \in T_{\beta_n+1}$.

Then set

$T_{\alpha+1} = \{\varphi(a_\alpha) \colon \varphi(v) \in L_{\mathcal{A}}(\{a_\beta ; \beta < \alpha\})$ and there is $\theta \in \Theta_\alpha$ s.t. $\bigwedge v(\theta \to \varphi) \in T_\alpha\}$.

Using induction on the T_α's, it is easy to see that

(1) $\alpha < \lambda \Rightarrow T_\alpha = T_\lambda \cap L_{\mathcal{A}}(\{a_\beta ; \beta < \alpha\})$

(2) $\varphi \in T_\lambda$, $\psi \in L_{\mathcal{A}}(\{a_\alpha ; \alpha < \lambda\})$, $\vdash \varphi \to \psi$ imply $\psi \in T_\lambda$

(3) $\perp \notin T_\lambda$

Claim. *The hypothesis of the Compactness Lemma II.5 is satisfied when* $\Gamma_0 = L_{\mathcal{A}}(\{a_\alpha ; \alpha < \lambda\})$ *and* $T = T_\lambda$.

Proof. Note that by (2) $\{\varphi\colon T_\lambda \vdash^* \varphi\}$ is T_λ itself. So we have to show that T_λ is closed under $\bigwedge\!\!\bigwedge$, which is done in the following way: writing $T^{\alpha_1 \cdots \alpha_n}$ for $T_\lambda \cap \mathrm{L}_{\mathscr{A}}(\{a_{\alpha_1} \dots a_{\alpha_n}\})$, where $\alpha_1 < \dots < \alpha_n < \lambda$, one shows by induction on α_n that $T^{\alpha_1 \cdots \alpha_n}$ is a Σ subset of $\mathrm{L}_{\mathscr{A}}(\{a_{\alpha_1} \dots a_{\alpha_n}\})$, closed under $\bigwedge\!\!\bigwedge$. So we assume that $T^{\beta_1 \cdots \beta_p}$ is Σ and closed under $\bigwedge\!\!\bigwedge$, whenever $\beta_1 < \dots < \beta_p < \alpha_n$. There exist $\beta_1 < \dots < \beta_p$ such that $\{\alpha_1, \dots, \alpha_{n-1}\} \subset \{\beta_1 \dots \beta_p\}$, $\beta_p < \alpha_n$, and $\Theta_{\alpha_n} \subset \mathrm{L}_{\mathscr{A}}(\{a_{\beta_1} \dots a_{\beta_p}\})$. Then if $\varphi(v) \in \mathrm{L}_{\mathscr{A}}(\{a_{\alpha_1} \dots a_{\alpha_{n-1}}\})$, because of (1) and of the definition of T_{α_n}
$\varphi(a_{\alpha_n}) \in T^{\alpha_1 \cdots \alpha_n} \Leftrightarrow (\exists \theta \in \Theta_{\alpha_n})[\bigwedge v(\theta \to \varphi) \in T^{\beta_1 \cdots \beta_p}]$.

Since Θ_{α_n} and (by the induction hypothesis) $T^{\beta_1 \cdots \beta_p}$ are Σ, this shows that $T^{\alpha_1 \cdots \alpha_n}$ is Σ.

Now consider $\bigwedge\!\!\bigwedge \Phi(v) \in \mathrm{L}_{\mathscr{A}}(\{a_{\alpha_1} \dots a_{\alpha_{n-1}}\})$ such that $\Phi(a_{\alpha_n}) \subset T^{\alpha_1 \cdots \alpha_n}$: if $F(\theta, \varphi)$ is a Σ formula expressing in $\mathscr{A}$ that $(\theta \in \Theta_{\alpha_n}) \wedge [\bigwedge v(\theta \to \varphi) \in T^{\beta_1 \cdots \beta_p}]$ then $(\forall \varphi \in \Phi)\, \exists \theta F(\theta, \varphi)$.

By Lemma I.2, there exists $\Psi \in \mathscr{A}$ such that $(\forall \varphi \in \Phi)(\exists \theta \in \Psi) F(\theta, \varphi)$ and $(\forall \theta \in \Psi)(\exists \varphi \in \Phi) F(\theta, \varphi)$: then $\Psi \subset \Theta_{\alpha_n}$, hence $\bigwedge\!\!\bigwedge \Psi \in \Theta_{\alpha_n}$, and $\bigwedge v(\bigwedge\!\!\bigwedge \Psi \to \bigwedge\!\!\bigwedge \Phi)$ belongs to $T^{\beta_1 \cdots \beta_p}$; so $\bigwedge\!\!\bigwedge \Phi(a_{\alpha_n}) \in T^{\alpha_1 \cdots \alpha_n}$.

We apply Lemma II.5 as indicated in the claim: since $\bot \notin T_\lambda$, T_λ is consistent. Moreover, by Lemma II.4, there is a model N of $\mathrm{L}_{\mathscr{A}}(\{a_\alpha;\ \alpha < \lambda\})$ such that $\{\varphi\colon T_\lambda \vdash^* \varphi\} = T_\lambda = \{\varphi\colon N \models \varphi\}$.

Clearly, we can assume that when we constructed the sequence of T_α's, we chose $\{\Theta_\alpha; \alpha < \lambda\}$ so that all sets Θ with the property (*) are enumerated, each with λ repetitions.

Let M be $N \upharpoonright \{a_\alpha; \alpha < \lambda\}$. We check that M is an $\mathrm{L}_{\mathscr{A}}$-submodel of N: if $\varphi(v) \in \mathrm{L}_{\mathscr{A}}(M)$, then $\Theta = \{\bigvee u \varphi(u) \to \varphi(v)\}$ satisfies (*), hence is equal to Θ_α for some α; then a_α is an element of M such that $N \models \bigvee u \varphi(u) \to \varphi(a_\alpha)$, so $\bigvee u \varphi(u)_N = \varphi(a_\alpha)_N$. By the Löwenheim–Skolem theorem I.1, $M \models \varphi \Leftrightarrow N \models \varphi \Leftrightarrow \varphi \in T_\lambda$, for any sentence $\varphi \in \mathrm{L}_{\mathscr{A}}(M)$.

From which follows, for any set Θ:
(*) holds $\Leftrightarrow$ Θ is a Σ type in M.

Then M is a Σ-saturated model:

$\{a_\alpha; \alpha < \lambda\}$ is an enumeration of $|M|$ such that a_α realizes a Σ type over $\{a_\beta; \beta < \alpha\}$ (namely Θ_α);

if $\Theta(v)$ is a Σ type in M – so satisfies (*) – and X is a subset of M of power $< \lambda$, there is $\gamma < \lambda$ such that $\Theta_\gamma = \Theta$ and $X \subset \{a_\alpha; \alpha < \gamma\}$. Then since a_γ realizes Θ_γ over $\{a_\alpha; \alpha < \gamma\}$, a_γ realizes Θ over X.

Definition. We say that a model M of L is a *Σ-model* if the theory realized by M in $\mathrm{L}_{\mathscr{A}}(\bar{a})$ is Σ, for every sequence $\bar{a}$ in $|M|$.

Proposition 2. (a) *Any Σ-saturated model is an $\mathrm{L}_{\mathcal{A}}$-model and a Σ-model*

(b) *For M denumerable, M is Σ-saturated if and only if:*
(2.i) *M is a Σ-model, and for every Σ type $p(v)$ and every finite subset X of M, there is $a \in M$ which realizes p over X*

(c) *If M is denumerable and Σ-saturated, $\psi(v) \in \mathrm{L}_{\mathcal{A}}$, $\mathrm{L}^0 \subset \mathrm{L}$, then $M \restriction \mathrm{L}^0$ and $M \restriction \psi$ $(= M \restriction \{a \in |M| : M \models \psi(a)\})$ are Σ-saturated.*

Proof. (a) If M is Σ-saturated, and if $\psi(v) \in \mathrm{L}_{\mathcal{A}}(M)$, then since $\{\mathbf{V} u\, \psi(u) \to \psi(v)\}$ is a Σ type in M, there exists $a \in M$ realizing it; so $\psi(a)_M = \mathbf{V} u\, \psi(u)_M$, hence M is an $\mathrm{L}_{\mathcal{A}}$-model.

To show that M is also a Σ-model, only Cond. (1.i) is needed. Let $\{a_\alpha ; \alpha < \gamma\}$ be the enumeration of $|M|$ given by (1.i); we set $T^{\alpha_1 \dots \alpha_n} = \{\theta \in \mathrm{L}_{\mathcal{A}}(\{a_{\alpha_1}, ..., a_{\alpha_n}\}) : M \models \theta\}$ and we can prove that $T^{\alpha_1 \dots \alpha_n}$ is Σ, very much as we proved the same statement in the Claim of Proposition 1.

(b) follows easily from (a).

(c) follows easily from the definition (2.i) of Σ-saturation.

Uniqueness result. We want to extend to Σ-saturated models the uniqueness property which holds for saturated models. Clearly some conditions must be added to (1.i) and (1.ii) to ensure uniqueness in a strict sense: for given a Σ-saturated model M, we see that

(a) by embedding the boolean algebra of M into a larger one it is easy to obtain a non isomorphic model realizing the same theory, which still satisfies (1.i) and (1.ii), and that

(b) trivial variants of M arise because we did not restrict ourselves to equality models.

If N is a Σ-saturated model with boolean algebra $\mathbb{B}$, let $\mathbb{B}_1$ be the restriction of $\overline{\mathbb{B}}$ to $\{\theta_N : \theta$ a sentence of $\mathrm{L}_{\mathcal{A}}(N)\}$, and let M be the model obtained by considering N as a $\mathbb{B}_1$-valued model. Clearly, for any $\psi \in \mathrm{L}_{\mathcal{A}}(M) = \mathrm{L}_{\mathcal{A}}(N)$, $\psi_M = \psi_N$. So M remains a Σ-saturated model realizing the same theory; in addition it satisfies

Condition 3.i. *The domain of the boolean algebra of M is equal to the set of all elements of the form θ_M for a sentence $\theta \in \mathrm{L}_{\mathcal{A}}(M)$.*

We shall avoid (a) by considering only Σ-saturated models M which satisfy Condition 3.i; it is also easy to avoid (b): we could require that

M is an equality model, as defined on p. 45; but we shall prefer an opposite solution. We consider the

Condition 3.ii. *If* $a \in M$, *then* $\{b\colon M \models a = b\}$ *is of power* $\|M\|$.

Proposition 3. *Let T be a Σ theory in* $L_{\mathscr{A}}$. *In every regular cardinal* λ *there exists a model realizing T, which is Σ-saturated and satisfies* (3.i), (3.ii); *this model is unique up to isomorphism.*

Proof of existence. It is clear that any Σ-saturated model M can be made to satisfy (3.i) and (3.ii), in such a way that the existence part of the proposition follows from Proposition 1.

Proof of uniqueness. We prove uniqueness only for denumerable models (we shall not use Σ-saturated models of higher power). So let M, N be Σ-saturated denumerable models realizing the same theory. We say that a partial mapping $f\colon M \to N$ *is an* $L_{\mathscr{A}}$*-morphism* if for every sentence ψ of $L_{\mathscr{A}}(\operatorname{dom} f)$, $M \models \psi \Leftrightarrow N \models f\psi$. By induction on n, we define enumerations $\{a_n\,; n < \omega\}$ of $|M|$ and $\{b_n\,; n < \omega\}$ of $|N|$, such that the map $f\colon a_n \to b_n$ $(n < \omega)$ is an $L_{\mathscr{A}}$-morphism. Then by Condition 3.i, the hypothesis of Lemma II.C holds, with Γ equal to $L_{\mathscr{A}}$. By this lemma, f will be a relative isomorphism between M and N.

Our induction hypothesis is that $\bar{a}, \bar{b}$ are chosen and that the map $f_n\colon a_i \to b_i$ $(i < n)$ is an $L_{\mathscr{A}}$-morphism. This hypothesis is true initially because M and N realize the same theory. We assume it now for an arbitrary n. Then if $n = 2k$ we choose for a_n the $(k+1)^{\text{th}}$ element of a fixed ω-sequence enumerating $|M|$; by Proposition 2.a, a_n realizes a Σ type $p(v)$ over $\bar{a}$; if $\theta \in p$ then $M \models \bigvee v\theta$ hence by induction hypothesis $N \models \bigvee v f_n \theta$. So $f_n p$ is a Σ type in N, and since N is Σ-saturated we can choose for b_n an element which realizes $f_n p$ over $\bar{b}$. Applying again the induction hypothesis, we see that in this way f_{n+1} is an $L_{\mathscr{A}}$-morphism.

If $n = 2k+1$ we choose for b_n the $(1+k)^{\text{th}}$ element of a fixed ω-sequence enumerating $|N|$, and then we choose a_n in M so that f_{n+1} becomes an $L_{\mathscr{A}}$-morphism – the existence of a_n is shown as was the existence of b_n in the even case.

Throughout the rest of this paper, we assume that Σ-saturated models are denumerable and satisfy Conditions 3.i *and* 3.ii, *so that we can always apply the uniqueness result.*

Remark 4. With some small modifications, *Proposition 2 remains true*

for this restricted notion of Σ-saturation. In particular 2.c becomes:
if M is Σ-saturated and satisfies $\mathsf{V}x\mathsf{V}y\ (x \neq y)$ (resp.: $\mathsf{V}x\mathsf{V}y\ [(x \neq y) \wedge \psi(x) \wedge \psi(y)]$), then $M\restriction \mathrm{L}^0$ (respectively $M\restriction\psi$) is Σ-saturated.

Proof. If M is Σ-saturated then $M\restriction\mathrm{L}^0$ still satisfies (3.ii) and (by Proposition 2.c) (1.i), (1.ii). To show that $M\restriction\mathrm{L}^0$ satisfies (3.i), hence is Σ-saturated, consider an element of the boolean algebra of M; it is of the form θ_M for some sentence θ of $\mathrm{L}_{\mathscr{A}}(M)$. Since $\mathsf{V}x\mathsf{V}y\ [(x = y) \leftrightarrow \theta]$ is consequence of $\mathsf{V}x\mathsf{V}y\ (x \neq y)$, it holds in M and since M is an $\mathrm{L}_{\mathscr{A}}$-model there exist a, a' in M such that $M \models (a = a') \leftrightarrow \theta$. We thus showed that θ_M is also of the form θ'_M for some sentence θ' of $\mathrm{L}^0{}_{\mathscr{A}}\ (M)$, namely for $\theta' = (a = a')$. This ends the proof that $M\restriction\mathrm{L}^0$ is Σ-saturated; and the proof for $M\restriction\psi$ is similar.

§IV. Chains of Σ-saturated models

In this paragraph, we prove the new results mentioned in the introduction. In addition, we use one of them to construct particular "short uncountable models" of ZF, in the sense of Keisler [3] (see Proposition 10).

We assume here that $\mathcal{A}$ is a denumerable admissible set, satisfying the well ordering axiom: in $\mathcal{A}$, every set is bijectable on an ordinal. L shall be a language of $\mathcal{A}$. The proposition 3 sums up the construction of the various chains of Σ-saturated models we shall use in this paragraph. First we need two lemmas.

Lemma 1. *Suppose μ is an ordinal of $\mathcal{A}$, and $\mathcal{A}$ satisfies: "every set is of power $\leq \mu$". We consider a formula $\bigwedge\!\!\bigwedge\{R_\alpha(\bar{v}); \alpha < \mu\} \in \mathrm{L}_{\mathcal{A}}$, and set $F_\mu = [\bigvee \bar{v} \bigwedge\!\!\bigwedge_{\alpha<\mu} R_\alpha(\bar{v})] \wedge [\bigwedge\!\!\bigwedge_{\alpha<\mu} (\bigvee \bar{v} \bigwedge\!\!\bigwedge_{\beta<\alpha} R_\beta(\bar{v}) \wedge \neg R_\alpha(\bar{v})]$; Γ denotes a subset of $\mathrm{L}_{\mathcal{A}}$ containing $\bigwedge\!\!\bigwedge_{\beta<\alpha} R_\beta$ for any $\alpha < \mu$. If M is an $\mathrm{L}_{\mathcal{A}}$-model of F_μ and $M/\bar{D}$ is a Γ-quotient of M, then $M/\bar{D}$ is an $\mathrm{L}_{\mathcal{A}}$-quotient of M.*

Proof. By Lemma II.2 it is enough to consider $\bigwedge\!\!\bigwedge \Phi(\bar{v}) \in \mathrm{L}_{\mathcal{A}}$ and $\bar{a} \in M$ such that $\{\varphi(\bar{a})_M : \varphi \in \Phi\} \subset D$, and to show $\bigwedge\!\!\bigwedge \Phi(\bar{a})_M \in D$. We may assume $\Phi = \{\varphi_\alpha; \alpha < \mu\}$. The following formula is consequence of F_μ: $\bigwedge \bar{x} \bigvee \bar{v} \bigwedge\!\!\bigwedge_{\alpha<\mu} [(\bigwedge\!\!\bigwedge_{\beta<\alpha} \varphi_\beta(\bar{x})) \leftrightarrow (\bigwedge\!\!\bigwedge_{\beta<\alpha} R_\beta(\bar{v}))]$; M being an $\mathrm{L}_{\mathcal{A}}$-model, there is $\bar{b}$ such that for all $\alpha < \mu$ $(\bigwedge\!\!\bigwedge_{\beta<\alpha} \varphi_\beta(\bar{a}))_M = (\bigwedge\!\!\bigwedge_{\beta<\alpha} R_\beta(\bar{b}))_M$. Then, by induction on $\alpha \leq \mu$ (using the hypothesis on D), for all α $(\bigwedge\!\!\bigwedge_{\beta<\alpha} R_\beta(\bar{b}))_M \in D$, so $(\bigwedge\!\!\bigwedge \Phi(\bar{a}))_M \in D$.

We fix now a trivial way to extend any model of L in such a way that for some set of formulas $\{R_\alpha; \alpha \leq \mu\}$, it will satisfy the formula F_μ of Lemma 1.

For every model A of L we denote by A^* the model obtained by adding to the domain of A new and distinct elements $\{c_\alpha; \alpha \leq \mu\}$ and by requiring for every sentence θ of $\mathrm{L}_{\mathcal{A}}(A^*)$:

$\theta_{A^*} = \theta_A$ if θ_A was already defined; $\theta_{A^*} = 1$ if θ is $c_\alpha = c_\alpha$ and $\theta_{A^*} = 0$ in any other case.

We let L* be the language L with $\{c_\alpha; \alpha \leq \mu\}$ as a set of new constants.

Lemma 2. *If A is an $\mathrm{L}_{\mathcal{A}}$-submodel of B then A^* is an $\mathrm{L}^*_{\mathcal{A}}$-submodel of B^*.*

Proof. There is a trivial way to associate to every formula $\varphi(\bar{v})$ of $L^*_{\mathscr{A}}$ a formula $\psi(\bar{v})$ of $L_{\mathscr{A}}$ such that for every model A_1 of L, $\varphi_{A_1} = \psi_{A_1^*} \restriction |A_1|$. Then if A is an $L_{\mathscr{A}}$-submodel of B and $\bar{a} \in A$, $A^* \models \varphi(\bar{a}) \Leftrightarrow A \models \psi(\bar{a}) \Leftrightarrow$ $\Leftrightarrow B \models \psi(\bar{a}) \Leftrightarrow B^* \models \varphi(\bar{a})$.

Proposition 3. *Let* L′ *be a language of* $\mathscr{A}$ *extending* L *and let* R *be a one-place relation of* L′−L. *Suppose* A_1 *is a model of* L′ *such that the model* $A_1 \restriction R$ *is an* $L_{\mathscr{A}}$*-submodel of* A_1. *Then for any* Σ *subset* Φ *of* $L'_{\mathscr{A}}$ *true in* A_1, *there exists* $\{B'_\alpha ; \alpha < \omega_1\}$, $L_{\mathscr{A}}$*-elementary chain of* $\mathbb{2}$*-models of* Φ *such that for any* $\alpha < \omega_1$, $R_{B'_{\alpha+1}} = |B'_\alpha|$.

Proof. We shall assume there is an ordinal $\mu \in \mathscr{A}$ such that $\mathscr{A}$ satisfies "every set is of power $\leq \mu$"; but it is easy to extend the proof to sets $\mathscr{A}$ which satisfy simply the well-ordering axiom. Let A_1, Φ satisfy the hypothesis.

We assume that L contains a set of constants, $\{c_\alpha ; \alpha \leq \mu\}$, and that $\alpha < \beta \leq \mu \Rightarrow A_1 \models c_\alpha \neq c_\beta$, since Lemma 2 enables us to reduce to this case, by considering A_1^* instead of A_1 and $R(v) \vee \mathbf{W}\{v = c_\alpha ; \alpha \leq \mu\}$ instead of $R(v)$.

For any formulas $\varphi(v)$ and θ, $\theta^{(\varphi)}$ will denote the relativization to φ of the formula θ. We extend Φ so that it contains
$\bigwedge \bar{x} R(x_0) \wedge ... \wedge R(x_{n-1}) \to (\theta^{(R)}(\bar{x}) \leftrightarrow \theta(\bar{x}))$, for all $\theta(\bar{x}) \in L_{\mathscr{A}}$.
By induction on α, we construct an $L_{\mathscr{A}}$-elementary chain of Σ-saturated models $\{B_\alpha ; \alpha < \omega_1\}$, such that for all $\alpha, B_{\alpha+1}$ can be enriched in a model $B'_{\alpha+1}$ of L′, which is a model of Φ such that $R_{B'_{\alpha+1}} = |B_\alpha|$:

we let B'_1 be a Σ-saturated model of Φ, and set $B_1 = B'_1 \restriction L$, $B_0 = (B'_1 \restriction R) \restriction L$;

if $\alpha = \beta + 1$ and B_β is constructed, then since B_β is Σ-saturated and $L_{\mathscr{A}}$-equivalent to B_0, B_β is relatively isomorphic to B_0 (by Prop. III.2.c, III.3 and Remark III.4); as there exists B'_1, Σ-saturated model of Φ such that $(B'_1 \restriction R) \restriction L = B_0$, there also exists B'_α, Σ-saturated model of Φ such that $(B'_\alpha \restriction R) \restriction L = B_\beta$, and we can set $B_\alpha = B'_\alpha \restriction L$;

if α is a limit ordinal, and B_β is defined for $\beta < \alpha$, then using Proposition III.2.b and Remark III.4 it is easy to see that $\bigcup_{\beta<\alpha} B_\beta$ is Σ-saturated and we set $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$.

Let $\mathbb{B}$ be the algebra of values of B'_1, and D be an ultrafilter in $\mathbb{B}$, such that B'_1/D is an $L'_{\mathscr{A}}$-quotient of B'_1.

Lemma. *If* $\alpha < \omega_1$, $B'_{\alpha+1}/D$ *is an* $L'_{\mathscr{A}}$*-quotient of* $B'_{\alpha+1}$.

Proof. Let Γ be the set of formulas of $L_{\mathscr{A}}$ that are relativised to $(\mathbf{W}_{\alpha\leq\mu}\, v = c_\alpha)$. Since Φ implies $\mathbf{\Lambda} v(\mathbf{W}_{\alpha\leq\mu}\, v = c_\alpha \to R(v))$, an induction on $\alpha < \omega_1$ proves that these formulas have the same interpretation in $B'_{\alpha+1}$ and B'_1, and so $B'_{\alpha+1}/D$ is a Γ-quotient of $B'_{\alpha+1}$. Now apply Lemma 1 with $M = B'_{\alpha+1}$ and, for $\alpha < \mu$, $R_\alpha(v) = [\mathbf{W}_{\alpha\leq\beta\leq\mu}\, v = c_\beta]$; $B'_{\alpha+1}/D$ is an $L'_{\mathscr{A}}$-quotient of $B'_{\alpha+1}$.

From the Lemma it follows that the properties of the chain $\{B'_\alpha;\ \alpha < \omega_1\}$ are preserved by the quotients; that is $\{B'_\alpha/D; \alpha < \omega_1\}$ is a chain of $\mathfrak{2}$-models satisfying the conclusion of the proposition.

Reflection axioms. This part does not use the well-ordering axiom for $\mathscr{A}$. We assume that L contains a binary relation E.

Lemma 4. *There exists T_0 and T_1, theories which are Δ in $L_{\mathscr{A}}$ such that:*
if A_1 has a proper $L_{\mathscr{A}}$-submodel then $A_1 \models T_0$; the converse holds if A_1 is Σ-saturated; and
if A_1 has a proper transitive (for E_{A_1}) $L_{\mathscr{A}}$-submodel, then $A_1 \models T_1$; the converse holds if A_1 is Σ-saturated.

Notations. Let $U = \{u_n; n < \omega\}$ be a set of variables with an infinite complement. $L^U_{\mathscr{A}}$ is the set of formulas of $L_{\mathscr{A}}$ whose free variables are among U, and whose bound variables are outside of U.

Let Γ^0 (resp. Γ^1) be the closure of $L^U_{\mathscr{A}}$ in $L_{\mathscr{A}}$, under conjunctions and quantifications of the form $\mathbf{V} u$ (resp. $\mathbf{V} u$ and $(\mathbf{\Lambda} u E v)$), u any variable of U. One defines T_0 (resp. T_1) as the collection of formulas $\varphi \to \mathbf{V} x \varphi^*(x)$ for all $\varphi \in \Gamma^0$ (resp. Γ^1), $\varphi^*(\bar{u}x)$ being the formula obtained by substituting $(\mathbf{V} u,\ u \neq x)$ for $\mathbf{V} u$ in $\varphi(\bar{u})$, for all $u \in U$.

Proof of Lemma 4 for T_0. Suppose A_0 is a proper $L_{\mathscr{A}}$-submodel of A_1; by induction on $\varphi(\bar{u}) \in \Gamma^0$, one sees that for all $\bar{a}$ in A_0 and b in $|A_1| - |A_0|$, $A_1 \models \varphi(\bar{a}) \to \varphi^*(\bar{a}b)$: so $A_1 \models T_0$.

Conversely let A_1 be a Σ-saturated model of T_0. The set of all formulas $\varphi^*(x)$, φ a valid sentence of Γ^0, is a Σ type in A_1, and we take $b \in A_1$ satisfying it. Let $\{F_n(\bar{u}u_n); n < \omega\}$ enumerate all formulas of $L^U_{\mathscr{A}}$ of the form $\mathbf{V} v \varphi(\bar{u} v) \to \varphi(\bar{u} u_n)$. We construct by induction on n a sequence $\{a_n; n < \omega\} \subset |A_1| - \{b\}$ such that for all $n < \omega$ and $\varphi(\bar{u} u_n) \in \Gamma$, A_1 satisfies $F_n(\bar{a}a_n)$ and $\vdash (F_0 \wedge ... \wedge F_n) \to \varphi$ implies $A_1 \models \varphi^*(\bar{a} a_n b)$. Then $A_0 = A_1 \restriction \{a_n; n < \omega\}$ will be a proper $L_{\mathscr{A}}$-submodel of A_1, as wanted: for if $\varphi(v) \in L_{\mathscr{A}}(A_0)$, there exists n such that

$F_n(\bar{a}u_n)$ is equivalent to $\mathbf{V}v\varphi(v) \to \varphi(u_n)$; so a_n is an element of A_0 such that $\varphi(a_n)_{A_1} = \mathbf{V}v\varphi(v)_{A_1}$.

So assume $\bar{a}$ constructed; it is enough to show that $p(u_n) = \{u_n \neq b \wedge F_n(\bar{a}u_n) \wedge \varphi^*(\bar{a}u_n b) : \vdash F_0 \wedge \ldots \wedge F_n \to \varphi(\bar{u}u_n)$, $\varphi(\bar{u}u_n) \in \mathrm{L}^U_{\mathscr{A}}\}$ is a Σ type in A_1; for then, one takes a_n satisfying p in A_1. Now p is closed under $\mathbf{\Lambda\!\!\Lambda}$ because $(\mathbf{\Lambda\!\!\Lambda}\Phi)^*(x) = \mathbf{\Lambda\!\!\Lambda}_{\varphi\in\Phi}\varphi^*(x)$, and because if for all $\varphi \in \Phi(\bar{u}u_n) \vdash F_0 \wedge \ldots \wedge F_n \to \varphi$, then $\vdash (F_0 \wedge \ldots \wedge F_n \to \mathbf{\Lambda\!\!\Lambda}\Phi)$. Moreover every formula of p is satisfiable in A_1: $\vdash (F_0 \wedge \ldots \wedge F_n \to \varphi(\bar{u}u_n))$ implies $\vdash F_0 \wedge \ldots \wedge F_n \to \mathbf{V}u_n(F_n \wedge \varphi)$, so that by induction hypothesis $A_1 \models [\mathbf{V}u_n(F_n \wedge \varphi)]^*(\bar{a}b)$; that is, $A_1 \models \mathbf{V}u_n[u_n \neq b \wedge \wedge F_n(\bar{a}u_n) \wedge \varphi^*(\bar{a}u_n b)]$.

We omit the proof for T_1, as it is quite similar to the proof we give of the Lemma 5.

Given $\psi(u) \in \mathrm{L}^U_{\mathscr{A}}$, we denote by Γ_ψ the closure of $\mathrm{L}^U_{\mathscr{A}}$ under $\mathbf{\Lambda\!\!\Lambda}$ and quantifications of the form $\mathbf{V}u$ and $\mathbf{\Lambda}u(\psi(u) \to \ldots)(u \in U)$. If $\varphi(\bar{u}) \in \Gamma_\psi$, $\varphi^*(\bar{u}x)$ is obtained by substituting $(\mathbf{V}u,\ u \neq x)$ for $\mathbf{V}u$ in φ (for all $u \in U$).

Lemma 5. *Let T_ψ be the set of all formulas $\varphi \to \mathbf{V}x\varphi^*(x)$, φ a sentence of Γ_ψ. If A_1 has a proper $\mathrm{L}_{\mathscr{A}}$-submodel A_0 such that $\psi_{A_1} \subset |A_0|$, then $A_1 \models T_\psi$; the converse holds if A_1 is Σ-saturated.*

Proof. To prove the first part of the lemma, consider A_1, A_0 satisfying its hypothesis. By induction on $\varphi(\bar{u}) \in \Gamma_\psi$, one sees that for all $\bar{a}$ in A_0 and all b in $|A_1| - |A_0|$, $A_1 \models \varphi(\bar{a}) \to \varphi^*(\bar{a}b)$. In particular $A_1 \models T_\psi$.

To show the converse, consider A_1, a Σ-saturated model of T_1. We may assume that $A_1 \models \mathbf{V}v\psi(v)$, otherwise the proof reduces to the proof of Lemma 4 for T_0.

Then we make the same construction of b and of $\{a_n ; n < \omega\} = |A_0|$ as we did in the proof of Lemma 4 for T_0, keeping also the same notations – *except that we everywhere replace Γ_0 by Γ_ψ.*

Claim. *Suppose $F_n(\bar{u}u_n)$ equivalent to $\psi(u_n)$; then if $\bar{a}$ satisfies the induction hypothesis (of the construction of $|A_0|$), $\bar{a}a_n$ satisfies it as soon as $A_1 \models \psi(a_n)$.*

For assume that F_n, $\bar{a}$ have these properties and $A_1 \models \psi(a_n)$. Given any formula $\varphi(\bar{u}u_n)$ of Γ_ψ such that $\vdash F_0 \wedge \ldots \wedge F_n \to \varphi(\bar{u}u_n)$, we have to show $A_1 \models \varphi^*(\bar{a}a_n b)$. Indeed, $\vdash F_0 \wedge \ldots \wedge F_{n-1} \to \mathbf{\Lambda}u_n[\psi(u_n) \to \varphi(\bar{u}u_n)]$, and by the induction hypothesis on $\bar{a}$, $A_1 \models [\mathbf{\Lambda}u_n\psi(u_n) \to \varphi]^*(\bar{a}$

that is $A_1 \models \bigwedge u_n [\psi(u_n) \to \varphi^*(\bar{a} u_n b)]$; hence $A_1 \models \varphi^*(\bar{a} a_n b)$. This establishes the Claim.

This freedom in the choice of a_n given by the claim can be used during the construction of $\{a_n ; n < \omega\}$ to assure that $\{a : A_1 \models \psi(a)\} \subset |A_0|$. Then A_0 is a submodel of A_1 with the desired properties.

Upward Löwenheim–Skolem and completeness results

Proposition 7. *Let Φ_0 be a Σ subset of $\mathrm{L}_{\mathscr{A}}$.* (a) *Φ_0 has a non denumerable model if and only if it has a model A_1 with a proper $\mathrm{L}_{\mathscr{A}}$-submodel A_0.*

(b) *Φ_0 has a non denumerable model if and only if $T_0 \cup \Phi_0$ is consistent. Hence the completeness result: $\varphi \in \mathrm{L}_{\mathscr{A}}$ is true in all non denumerable models iff $T_0 \vdash \varphi$.*

Proof. (a) If A_1 is a model of Φ_0 with a proper $\mathrm{L}_{\mathscr{A}}$-submodel A_0, apply Proposition 3 with $\Phi = \Phi_0$ to get a non denumerable model of Φ_0; the converse is the downward Löwenheim–Skolem theorem.

(b) T_0 is true in all non denumerable models, by Lemma 4 and the Löwenheim–Skolem theorem. Conversely, if $\Phi_0 \cup T_0$ is consistent, it has a Σ-saturated model A_1, which by Lemma 4 has a proper $\mathrm{L}_{\mathscr{A}}$-submodel. Then by (a) Φ_0 has a non denumerable model.

The next two results have similar proofs.

Proposition 8. *Let Φ_0 be a Σ set of sentences of $\mathrm{L}_{\mathscr{A}}$, and $\psi(v) \in \mathrm{L}_{\mathscr{A}}$. The following are equivalent:*

(a) *Φ_0 has an (ω_1, ω)-model (= model of power ω_1 in which the interpretation of ψ is denumerable);*

(b) *Φ_0 has a model A_1 with a proper $\mathrm{L}_{\mathscr{A}}$-submodel A_0 such that $\psi_{A_1} \subset |A_0|$;*

(c) *$T_\psi \cup \Phi_0$ is consistent.*

In particular $\varphi \in \mathrm{L}_{\mathscr{A}}$ is true in all (ω_1, ω)-models iff $T_\psi \vdash \varphi$.

Proposition 9. *Assume that* L *contains a binary relation E. We call ω_1^*-model any model A of power ω_1 such that for all $a \in A$ the transitive closure of $\{a\}$ for E_A is denumerable. Let Φ_0 be a Σ set of sentences of $\mathrm{L}_{\mathscr{A}}$. Following are equivalent:* (a) *Φ_0 has an ω_1^*-model;* (b) *Φ_0 has a model A_1 with a proper transitive (for E_{A_1}) $\mathrm{L}_{\mathscr{A}}$-submodel A_0;* (c) *$T_1 \cup \Phi_0$ is consistent. In particular $\varphi \in \mathrm{L}_{\mathscr{A}}$ is true in all ω_1^*-models $\Leftrightarrow T_1 \vdash \varphi$.*

Application to transitive models of ZF. Let $\mathcal{M}$ be a transitive denumerable model of set theory. We recall briefly from [15] the construction of a boolean extension of $\mathcal{M}$, which is a model of set theory but not of AC: inside $\mathcal{M}$, we define

$\mathbb{B}$,$\mathbb{B}'$, the boolean algebras of regular open sets of $(2^\omega)^{\aleph_1}$ and $(2^\omega)^{\aleph_1+1}$ respectively;

the boolean extension of $\mathcal{M}$, $\mathbf{V}^{(\mathbb{B})}$;

a subcollection of $\mathbf{V}^{(\mathbb{B})}$: the "sets hereditarily definable in $\mathbf{V}^{(\mathbb{B})}$, by formulas of set theory admitting real numbers and elements of $\mathcal{M}$ as parameters".

We denote by $\mathcal{N}$ the restriction of $\mathbf{V}^{(\mathbb{B})}$ to this collection. $\mathcal{N}$ is an inner extension of $\mathcal{M}$ (that is $\mathcal{N}$ has the same ordinals as $\mathcal{M}$ and $\mathcal{M}$ is a transitive submodel of $\mathcal{N}$); it is a model of ZF + $\neg$ AC.

Proposition 10 [4]. *Let* L *be the language having* $\in$ *as relation, and* m *as constant for any* $m \in \mathcal{M}$. *For any* $\psi \in \mathrm{L}_{\omega_1}$ *which is true in* $\mathcal{N}$, $\mathcal{M}$ *has an inner extension* $\mathcal{M}'$, *of power* ω_1, *which is a* $\mathbf{2}$*-valued model of* ZF + ψ.

Proof. Let $\mathcal{N}'$ be the model defined as $\mathcal{N}$, but with $\mathbb{B}'$ in place of $\mathbb{B}$. By the methods of [15] it is easy to show:

$\mathcal{N}$ is a submodel of $\mathcal{N}'$, proper in the sense that $\mathcal{N}'$ contains $\boldsymbol{b}$ such that for all a in $\mathcal{N}$, $\mathcal{N}' \models a \neq b$;

for any $\bar{a}$ in $\mathcal{M}$ there exists a *relative* isomorphism from $\mathcal{N}$ onto $\mathcal{N}'$ which is the identity over $\mathcal{M} \cup \{\bar{a}\}$.

These facts imply that $\mathcal{N}$ is a proper L_{ω_1}-submodel of $\mathcal{N}'$; from which by Proposition 7.(a) we infer that any $\psi \in \mathrm{L}_{\omega_1}$ true in $\mathcal{N}$ has a $\mathbf{2}$-valued model $\mathcal{M}'$ of power ω_1. We can assume that ψ contains the conjunction of the following formulas of L_{ω_1}:

ZF: $\bigwedge x\,(x \text{ is an ordinal} \to \bigvee_{\alpha < \alpha_{\mathcal{M}}} x = \alpha)$ where $\alpha_{\mathcal{M}}$ is the collection of the ordinals of $\mathcal{M}$; $\bigwedge x\ [x \in m \leftrightarrow \bigvee_{a \in m} x = a]$, this for all $m \in \mathcal{M}$.

Then $\mathcal{M}'$ satisfies the required conditions.

Languages with the quantifier "there exists uncountably many". Let L be a language of $\mathcal{A}$. $\mathrm{L}_{\mathcal{A}}(\mathrm{Q})$ is the language defined as $\mathrm{L}_{\mathcal{A}}$, but with an

[4] In the sequel to this paper, we shall use the proof of this result to obtain a stronger statement: *Let* $\mathcal{A}$ *be a denumerable admissible set such that* $\mathcal{M} \in \mathcal{A}$ *and* $\mathcal{A}$ *satisfies the well-ordering axiom. Whenever G is an* $\mathcal{A}$*-generic ultra-filter over* $\mathbb{B}$, *then* $\mathcal{M}[\mathrm{G}]$ *has an* $\mathrm{L}_{\mathcal{A}}$ *-elementary inner extension of power* ω_1.

additional closure property: if $\psi \in L_{\mathscr{A}}(Q)$ and x is a variable, then $Qx\psi \in L_{\mathscr{A}}(Q)$.

We extend to $L_{\mathscr{A}}(Q)$ all syntactic and semantic notions and notations that we recalled or introduced in the preliminaries (free variables of a formula, model, interpretation ψ_M of a formula ψ in a model M, $\models$, $\vdash$), simply *by treating every formula* $Qx\psi$ *of* $L_{\mathscr{A}}(Q)$ *as an atomic formula* (*having as variables the free variables of* ψ *except* x), which we add to the true atomic formulas of L: thus a *model* M *of* $L_{\mathscr{A}}(Q)$ is a domain $|M|$ together with an interpretation: $c \to c_M$ of the constants of L, and with an interpretation of atomic sentences, which is a function: $\theta \to \theta_M$ satisfying $\theta(c)_M = \theta(c_M)_M$ for any constant c of L, *but this time, the domain of this function includes the sentences* $Qx\psi(x\bar{a})$ ($\bar{a} \in M$ and $\psi(x\bar{v}) \in L_{\mathscr{A}}(Q)$) in addition to the atomic sentences of $L_{\mathscr{A}}(M)$. Moreover, we shall say that a subset of $L_{\mathscr{A}}(Q)$ is *consistent* if it has a model *in this sense.*

But in addition to these definitions, we have that of a *standard* model M of $L_{\mathscr{A}}(Q)$: M is *standard* if it is a model of $L_{\mathscr{A}}(Q)$ in the above sense, if it is two-valued, and if for any $\bar{a} \in M$ and $\psi(\bar{u}x) \in L_{\mathscr{A}}(Q)$, $M \models Qx\psi(\bar{a}x)$ [$\psi(\bar{a}x)_M$ is non denumerable].

Keisler's completeness theorem. *Let* $\mathbf{A} = \{Q_i : i \leq 4\}$ *be the following set of axioms and schemes:*

Q_1 $\bigwedge x(\psi(x\bar{v}) \to \varphi(x\bar{v})) \to (Qx\psi(x\bar{v}) \to Qy\varphi(y\bar{v}))$

Q_2 $(Qx \bigvee \Phi) \to \bigvee \{Qx\varphi : \varphi \in \Phi\}$

Q_3 $\bigwedge v \neg Qx\ x = v$

Q_4 $Qx\bigvee y\varphi \to \bigvee y Qx\varphi \vee Qy \bigvee x\varphi$

Then a formula $\psi \in L_{\mathscr{A}}(Q)$ *is true in all standard models if and only if* $\mathbf{A} \vdash \psi$.

We shall write $Q^*\bar{x}\varphi$ for $\neg Qx_0 \dots Qx_{n-1} \neg\varphi$.

The theorem holds actually inside much smaller fragments than $L_{\mathscr{A}}(Q)$ (see [8, p. 60]) provided at least they are closed under Q-quantification. We now consider fragments $L_{\mathscr{A}}(\Phi) \subset L_{\mathscr{A}}(Q)$ which are not necessarily closed under Q-quantification, and shall prove a completeness result for them: for any subset Φ of $L_{\mathscr{A}}(Q)$, $L_{\mathscr{A}}(\Phi)$ is the set of formulas $\theta \in L_{\mathscr{A}}(Q)$ such that every subformula of θ of the form $Qx\psi$ is also subformula of an element of Φ.

We proceed as follows: we define a set $\mathbf{C}$ of easy consequences of $\mathbf{A}$ (Proposition 11) and prove (Proposition 12) that a Σ set Φ has a standard model if and only if it is consistent with $\mathbf{C} \cap \mathrm{L}_{\mathscr{A}}(\Phi)$ [5].

To compare **A** and **C**, one may say that the main axiom scheme of **A** is analogous to the "replacement axiom" of set theory, where as **C** is closer to the "reflection axiom", as appears in the proof of Lemma 12; but **C** has not a simple "axiom scheme" structure as **A**.

Proposition 11. *Let E be the set of formulas of $\mathrm{L}_{\mathscr{A}}(\mathrm{Q})$ defined by the following inductive conditions (these formulas have their free variables among $\{u_n ; n < \omega\}$ and $\{x_n ; n < \omega\}$):*

– $u_i \neq x_j \in E$, and E is closed under $\bigwedge\!\!\!\bigwedge$

– $\theta(\overline{u}\,\overline{x}) \in E$ implies $\bigvee u_i(F \wedge \theta) \in E$, for any $i < n$ and any $F \in \mathrm{L}_{\mathscr{A}}(\mathrm{Q})$ of the form: $[\bigvee u_i \varphi(\overline{u})] \to \varphi(\overline{u})$

– $\theta(\overline{u}\,\overline{x}) \in E$ implies $\mathrm{Q}y\,\psi(\overline{u}y) \to \bigvee x_i[\psi(\overline{u}x_i) \wedge \theta(\overline{u}\,\overline{x})] \in E$, for any $i < n$ and $\psi(\overline{u}y) \in \mathrm{L}_{\mathscr{A}}(\mathrm{Q})$

– $\theta(\overline{u}\,\overline{x}) \in E$ implies $\neg \mathrm{Q}u_i\,\psi(\overline{u}) \to \bigwedge u_i(\psi \to \theta) \in E$, for any $i < n$ and $\psi(\overline{u}) \in \mathrm{L}_{\mathscr{A}}(\mathrm{Q})$.

*Then for any $\theta(\overline{u}\,\overline{v}) \in E$, $\mathbf{A} \vdash \bigwedge \overline{u}\mathrm{Q}^*\overline{x}\,\theta(\overline{u}\,\overline{x})$. In particular, **C** being the set of all sentences of E, **C** is consequence of **A**.*

Proof. It is obvious, by induction on the clauses defining E, that for every standard model M and every $\theta(\overline{u}\overline{x}) \in E$, $M \models \bigwedge \overline{u}\mathrm{Q}^*\overline{x}\,\theta(\overline{u}\,\overline{x})$. So by Keisler's completeness theorem, $\mathbf{A} \vdash \bigwedge \overline{u}\mathrm{Q}^*\overline{x}\,\theta(\overline{u}\,\overline{x})$.

Proposition 12. *Let ψ be a formula of $\mathrm{L}_{\mathscr{A}}(\mathrm{Q})$. ψ is valid in all standard model if and only if $\mathbf{C} \cap \mathrm{L}_{\mathscr{A}}(\psi) \vdash \psi$. More generally, if Φ_0 is a Σ set of sentences of $\mathrm{L}_{\mathscr{A}}(\mathrm{Q})$, Φ_0 has a standard model if and only if Φ_0 is consistent with $\mathbf{C} \cap \mathrm{L}_{\mathscr{A}}(\Phi_0)$.*

Proof. We admit the following Lemma, since its proof is quite similar to that of Lemma 4 and 5:

Lemma 12. *Let A_1 be a Σ-saturated model, and suppose that there is a formula $\mathrm{Q}x\,\psi_0(\overline{u}x)$ of $\mathrm{L}_{\mathscr{A}}(\Phi_0)$ such that $A_1 \models \bigvee \overline{u}\mathrm{Q}x\,\psi_0(\overline{u}x)$; A_1 satisfies $\mathbf{C} \cap \mathrm{L}_{\mathscr{A}}(\Phi_0)$ if and only if there exists A_0, $\mathrm{L}_{\mathscr{A}}(\Phi_0)$-subnodel of A_1 such that:*

(*) for any formula $\mathrm{Q}x\,\psi(\overline{u}, x) \in \mathrm{L}_{\mathscr{A}}(\Phi_0)$, any $\overline{a} \in A_0$, $[\neg \mathrm{Q}x\,\psi(\overline{a}, x) \wedge \psi(\overline{a}\,v)]_{A_1} \subset \|A_0\|$, and $[\mathrm{Q}x\,\psi(\overline{a}, x) \to \psi(\overline{a}, v)]_{A_1} \not\subset |A_0|$.

[5] When $\Phi = \{\neg\psi\}$, this is exactly the "subformula property" of **C** mentioned in the introduction.

Now, the proof of the proposition is divided in two cases:

A) if $\neg \mathbf{V} \bar{u} Qx \psi_0(\bar{u}x)$ is consequence of $\Phi_0 \cup (\mathbf{C} \cap L_{\mathscr{A}}(\Phi_0))$ for every formula $Qx\psi_0(\bar{u}x)$ of $L_{\mathscr{A}}(\Phi_0)$, then any denumerable two-valued model of $\mathbf{C} \cap L_{\mathscr{A}}(\Phi_0) \cup \Phi_0$ will be a standard model of Φ_0.

B) if $\mathbf{C} \cap L_{\mathscr{A}}(\Phi_0)$, Φ_0 and $\mathbf{V} \bar{u} Qx \psi_0(\bar{u}x)$ have a model for some formula $Qx\psi_0(\bar{u}x)$ of $L_{\mathscr{A}}(\Phi_0)$, they have a Σ-saturated model, hence they have a model A_1 with a submodel A_0, which satisfy the conclusion of the Lemma. If we denote by L′ the language L with an additional one-place relation R, and we enrich A_1 to L′ by setting $R_{A_1} = |A_0|$, then the condition (*) on A_1, A_0 is expressed by a set Φ_1 which is a Σ theory in $L_{\mathscr{A}}(\Phi_0)$.

Prop. 3 applied to $\Phi = \Phi_0 \cup \Phi_1$ and A_1, when $L_{\mathscr{A}}$ is replaced by $L'_{\mathscr{A}}(\Phi_0)$, shows the existence of a chain $\{B_\alpha ; \alpha < \omega_1\}$ such that $\bigcup_{\alpha<\omega_1} B_\alpha$ is a standard model of Φ_0.

Remark 13. We can obtain Keisler's completeness theorem for $L_{\mathscr{A}}(Q)$ as an easy corollary to Proposition 12: because the completeness of **A** follows immediately from the completeness of **C** and from **A** $\vdash$ **C**. Actually, we used the completeness of **A** in our proof (Proposition 11) of **A** $\vdash$ **C**, but this is quite easy to avoid: one proceeds by the same induction as in Proposition 11, but gives a syntactic proof of each induction step.

Unions of chains. We call Σ_1 those formulas of $L_{\mathscr{A}}$ which are built from quantifier free formulas using only $\mathbf{V}$, $\mathbf{M}$, $\mathbf{W}$; and Π_2 those formulas of $L_{\mathscr{A}}$ built from Σ_1 formulas using only $\mathbf{\Lambda}$, $\mathbf{M}$ and $\mathbf{W}$. A classical result is:

(*) *a formula of* L_ω *is preserved by unions of arbitrary chains of models if and only if it is equivalent to a* Π_2 *formula of* L_ω.

The existence of analogs of (*) when $L_{\mathscr{A}}$ replaces L_ω depends on what notion of chain we consider:

(a) (Weinstein [16]). Preservation of Π_2 formulas under unions of chains of length α is not always true if $\alpha = \omega$; but is true, if $\alpha = \omega_1$. So we shall consider only chains of length ω_1.

(b) Even if we restrict ourselves to ω_1-chains, the extension to $L_{\mathscr{A}}$ of (*) is false: we give below a counter example, which answers a question of Weinstein [16].

(c) However, for some stronger notions of ω_1-chains, the analog of (*) holds; ω_1-chains of "*rank extensions*" (in the sense of set theory) would provide an example.

In spite of (b) we shall give a characterization result relative to preservation under unions of ω_1-chains, in the next paragraph. Here we give a semantic equivalent of this property, which allows us to replace consideration of ω_1-chains by that of particular chains of length 3, called sandwiches: (A_0, B, A_1) is a *sandwich* if $A_0 \subset B \subset A_1$ and A_0 is an $L_{\mathcal{A}}$-submodel of A_1.

Proposition 14. *For $\psi \in L_{\mathcal{A}}$, the following are equivalent:*
(a) *ψ is preserved under unions of ω_1-chains*
(b) *for every sandwich (A_0, B, A_1), $B \models \psi$ implies $A_1 \models \psi$.*

Proof. $\neg$(a) $\Rightarrow$ $\neg$(b). Let $\{B_\alpha; \alpha < \omega_1\}$ be an ω_1-chain of models of ψ whose union satisfies $\neg\psi$. $\bigcup_{\alpha<\omega_1} B_\alpha$ has a denumerable $L_{\mathcal{A}}$-submodel A_0; for some $\alpha < \omega_1$, $A_0 \subset B_\alpha$. $(A_0, B_\alpha, \bigcup_{\alpha<\omega_1} B_\alpha)$ shows $\neg$(b) to hold.

$\neg$(b) $\Rightarrow$ $\neg$(a). Let (A_0, B, A_1) be a sandwich such that $B \models \psi$ and $A_1 \models \neg\psi$. Let L$'$ be the language L $\cup$ $\{R, S\}$, where R, S are new unary relations. We enrich A_1 by setting $R_{A_1} = |A_0|$, $S_{A_1} = |B|$; then it becomes a model of $\Phi = \{\bigwedge v R(v) \to S(v), \neg\psi, \psi^{(S)}\}$. Applying Proposition 3, we get a chain $\{A'_\alpha; \alpha < \omega_1\}$ of $\mathbf{2}$-models of Φ. Then by Φ, for all $\alpha < \omega_1$ $B_\alpha = A'_{\alpha+1} \restriction S$ is a model of ψ, and A'_α an $L_{\mathcal{A}}$-submodel of $A'_{\alpha+1}$, which satisfies $\neg\psi$; so $\bigcup_{\alpha<\omega_1} B_\alpha \restriction L = \bigcup_{\alpha<\omega_1} A'_\alpha \restriction L$ satisfies $\neg\psi$: $\neg$(a) holds.

A counterexample. The formula

$$\psi = \bigvee x \bigwedge\!\!\!\bigwedge_{n<\omega} (\bigvee u \neq x)[\bigvee v_1 \bigwedge v_2 \bigvee v_3\, R_n(v_1 v_2 v_3) \to \bigwedge v_2 \bigvee v_3\, R_n(u v_2 v_3)]$$

(where for each n R_n is a relation symbol) is equivalent to a formula of T_0, hence (see Lemma 4) is true in all non denumerable models, which is stronger than being preserved under ω_1-chains. However, ψ is not equivalent to any Π_2 formula of L_{ω_1}. Indeed, for any denumerable admissible set $\mathcal{A}$, there exist models A, B such that (notations Σ_1, Π_2, being relative to $L_{\mathcal{A}}$) A is a Σ_1-submodel of B – which easily implies that all Π_2 formulas true in B hold in A – but $B \models \psi$ and $A \models \neg\psi$.

Construction of A and B. We obtain by induction a chain of denumerable models $\{A_n; n < \omega\}$, a chain of sets $\{E_n; n < \omega\}$, where E_n is a set of sentences of $L_\omega(A_n)$ which are true in A_n, and a sequence $\{a_n; n < \omega\}$, where $a_n \in A_n$:

suppose A_i, E_i, a_i chosen for $i < n$; then we choose a_n in $|A_n| - \{a_i, i < n\}$. Moreover,

– if $n = 3k$, we choose any sentence σ_k of $\Sigma_1(A_{n-1})$ (notation relative to $\mathscr{A}$) which is consistent with E_{n-1} and the (simple) diagram of A_{n-1}; A_n is then an extension of A_{n-1} satisfying E_{n-1} and σ_k, and $E_n = E_{n-1}$

– if $n = 3k + 1$, we choose $i < n$, $b^i_k \in A_{n-1} - \{a_i\}$, and take A_n with domain $|A_{n-1}| \cup \{c_k\}$, satisfying $E_n = E_{n-1} \cup \{\bigwedge y \neg R_i(b^i_k\, c_k\, y)\}$

– if $n = 3k + 2$, we choose $i < n$, $d^i_k \in A_{n-1}$, and take A_n with domain $|A_{n-1}| \cup \{e_k\}$ satisfying E_{n-1} and $R_i(a_i d^i_k\, e_k)$, and $E_n = E_{n-1}$. (If this were not possible, there should be a formula $\bigwedge y \neg R_i(a_i d^i_k\, y)$ in E_{n-1}; which is not allowed).

Let $A = \bigcup_{n<\omega} A_n$, $E = \bigcup_{n<\omega} E_n$. We may assume that the degrees of freedom in the above construction have been used to the effect that $\{\sigma_k ; k < \omega\}$ is an enumeration of all sentences of $\Sigma_1(A)$ which are consistent with the union of E and the diagram of A; that $\{a_n ; n < \omega\} = |A|$; that for each i, $\{b^i_k ; k < \omega\} = |A| - \{a_i\}$ and $\{d^i_k ; k < \omega\} = |A|$.

Then it is easy to check that $A \models \psi$ and that for any model B, $A \subset B$ implies A Σ_1-submodel of B. So we let B be any extension of A which satisfies E and is of power ω_1 (B exists by the compactness theorem for L_ω). Then $B \models \neg\, \psi$; so A, B are the required models.

§V. Σ-saturated models and projective relations

We illustrate the use of Σ-saturated models for characterization and interpolation results: we show that the theorem of Svenonius, [12], which relates local definability and preservation under automorphisms, is true in $L_{\mathscr{A}}$, provided one admits (relative) automorphisms of *boolean* models (Proposition 6); we give a proof of a result of Nebres, [4], relative to unions of "n-families of models"; and a proof of a somewhat refined version [6] of the interpolation theorem for many-sorted languages (Proposition 12). The theorem, due to Feferman, [1], shows that if one gives up interpolation of function symbols, one has a kind of interpolation for quantifiers: for any sort i of the language L, let us say that $\bigwedge^i$ (resp. $\bigvee^i$) *occurs in* a formula ψ if there is a variable u of sort i such that $\bigwedge u$ (resp. $\bigvee u$) occurs positively in ψ; then

$\bigwedge^i$ occurs in the interpolant only if $\bigwedge^i$ occurs in the premise, and $\bigvee^i$ occurs in the interpolant only if it occurs in the conclusion

is Feferman's interpolation clause.

We shall show that if in addition one restricts interpolation of constants, quantifiers can be interpolated in the same way as relations:

$\bigwedge^i$ occurs in the interpolant only if it occurs in the premise and the conclusion, and similarly for $\bigvee^i$.

We also prove some other interpolation results, especially for infinitary "Horn sentences" (Propositions 8 and 10). And prove a characterization result relative to preservation under unions of ω_1-chains (Proposition 7).

Characterization results. Let us make some general remarks on the uses of Σ-saturated models that we shall be making in the rest of the paragraph.

(A) Generally speaking, these methods provide simple proofs of characterization results for $L_{\mathscr{A}}$.

(B) These proofs have a common "converse" which is easy to obtain and quite general: we can interpret this converse as indicating that if a characterization result has a proof, then a priori it has one by means of the methods of (A).

We next give some details on (A) and (B), restricting ourselves for simplicity to characterization results of the type (**) below.

[6] This refinement was obtained by Stern and the author. For a proof based on "forcing", see [11]

(A) The uniqueness result in Proposition III.3 has generalizations of the following kind: one considers a relation $\mathcal{R}(M, N)$ on models and a syntactic relation $\psi \Gamma \varphi$ on formulas of $L_{\mathcal{A}}$; one then proves

(*) *Let M and N be Σ-saturated models such that whenever* $\psi \Gamma \varphi$ *then* $M \models \psi \Rightarrow N \models \varphi$ *and* $N \models \neg\varphi \Rightarrow M \models \neg\psi$; *then* $\mathcal{R}(M, N)$ *(and conversely).*

For the case where $\mathcal{R}(M, N)$ is the isomorphism relation and $\psi \Gamma \varphi$ when and only when ψ and φ are the same formula of $L_{\mathcal{A}}$, then (*) reduces to the uniqueness result of Proposition III.3 (for denumerable models). For general cases, we set some requirements on Γ:

Definition 1. $\psi \Gamma \varphi$ always denotes a Σ relation (on formulas $\psi, \varphi \in L_{\mathcal{A}}$ with the same free variables) which is *closed under* $\bigwedge$ *and* $\bigvee$; that is, $(\bigwedge_I \psi_i)\, \Gamma\, (\bigwedge_I \varphi_i)$ and $(\bigvee_I \psi_i)\, \Gamma\, (\bigvee_I \varphi_i)$, whenever $\psi_i \Gamma \varphi_i$ for all $i \in I$ and $\bigwedge_I \psi_i$, $\bigwedge_I \varphi_i$ belong to $L_{\mathcal{A}}$.

We are interested in (*) because it easily yields a characterization result:

(**) *Given* $\psi, \varphi \in L_{\mathcal{A}}$, *assume that* $M \models \psi$ *and* $\mathcal{R}(M, N)$ *imply* $N \models \varphi$. *Then there exists* ψ', φ' *such that* $\psi \vdash \psi'$, $\psi' \Gamma \varphi'$ *and* $\varphi' \vdash \varphi$ *(and conversely* [7]*).*

The proof of (*) ⇒ (**) is essentially contained in :

Lemma 1. Γ *satisfying the above requirements, let* Ψ, Φ *be* Σ *theories such that whenever* $\Psi \vdash \psi$ *and* $\psi \Gamma \varphi$, *then* φ *is consistent with* Φ. *If* $\Phi' = \{\varphi$: *there is* ψ *s.t.* $\Psi \vdash \psi$ *and* $\psi \Gamma \varphi\}$ *and* $\Psi' = \{\neg\psi$: *there is* φ *s.t.* $\Phi \cup \Phi' \vdash \neg\varphi$ *and* $\psi \Gamma \varphi\}$, *then* $\Phi \cup \Phi'$ *and* $\Psi \cup \Psi'$ *are consistent theories such that whenever* $\psi \Gamma \varphi$, *then* $\Psi \cup \Psi' \vdash \psi$ *implies* $\Phi \cup \Phi' \vdash \varphi$ *and* $\Phi \cup \Phi' \vdash \neg\varphi$ *implies* $\Psi \cup \Psi' \vdash \neg\psi$.

Proof. Since Γ is closed under $\bigwedge$, Φ' is closed under $\bigwedge$ so the compactness theorem implies the consistency of $\Phi \cup \Phi'$. Suppose $\neg\psi \in \Psi'$, and $\neg\psi$ is not consistent with Ψ. There is φ such that $\psi \Gamma \varphi$ and $\Phi \cup \Phi' \vdash \neg\varphi$; so there is ψ_0 such that $\Psi \vdash \psi_0$, $\psi_0 \Gamma \varphi_0$ and $\Phi \vdash \varphi_0 \to \neg\varphi$. Then $\Psi \vdash \psi_0 \wedge \psi$, $(\psi_0 \wedge \psi)\Gamma(\varphi_0 \wedge \varphi)$ and $\Phi \vdash \neg(\varphi_0 \wedge \varphi)$, which contradicts

[7] The converse is mostly very easy to prove, and we shall often neglect its proof.

the assumption of the lemma. So if $\neg\psi \in \Psi'$, $\neg\psi$ is consistent with Ψ. Since Γ is closed under $\mathbf{W}$, Ψ' is closed under $\mathbf{M}$ (up to equivalence) so by compactness $\Psi \cup \Psi'$ is consistent.

A similar argument proves that $\Psi \cup \Psi' \vdash \psi$ and $\psi\Gamma\varphi$ imply $\varphi \in \Phi'$. Then, since also $\Phi \cup \Phi' \vdash \neg\varphi$ and $\psi\Gamma\varphi$ imply $\neg\psi \in \Psi'$, the conclusion follows.

Now, to prove (*) $\Rightarrow$ (**), assume (*) and that ψ, φ do not satisfy the conclusion of (**): so if $\psi \vdash \psi'$ and $\psi'\Gamma\varphi'$ then φ' is consistent with $\neg\varphi$. Let Φ' and Ψ' be the sets defined in Lemma 1, when $\Psi = \{\psi\}$ and $\Phi = \{\neg\varphi\}$, and let M, N be Σ-saturated models realizing the theories $\{\psi\} \cup \Psi'$ and $\{\neg\varphi\} \cup \Phi'$. By Lemma 1, M and N satisfy the premise of (*); hence $\mathcal{R}(M, N)$. Since $M \models \psi$ and $N \models \neg\varphi$ this shows (the contrapositive of) (**).

We now end part (A) with a lemma and a remark which together with Lemma II.C allows us to prove the cases of (*) that are met in practice.

Definition. A function f is a Γ-*morphism*: $M \to N$ if $\operatorname{dom} f \subset |M|$, $\operatorname{im} f \subset |N|$ and whenever $\bar{a} \in \operatorname{dom} f$ and $\psi\Gamma\varphi$, then $M \models \psi(\bar{a}) \Rightarrow N \models f\varphi(\bar{a})$ and $N \models \neg f\varphi(\bar{a}) \Rightarrow M \models \neg\psi(\bar{a})$.

By extension, we shall say that $\emptyset$ *is a* Γ-*morphism*: $M \to N$ if we have whenever ψ, φ are sentences of $L_{\mathcal{A}}$ and $\psi\Gamma\varphi$, then $M \models \psi \Rightarrow N \models \varphi$ and $N \models \neg\varphi \Rightarrow M \models \neg\psi$.

We shall use Γ-morphisms also in cases where Γ is a subset of $L_{\mathcal{A}}$ (instead of $L_{\mathcal{A}} \times L_{\mathcal{A}}$); then it is assumed that in the above definitions $\psi\Gamma\varphi$ reduces to "$\psi \in \Gamma$ and $\varphi = \psi$".

Examples. Cond. (C) of the beginning of § II says that f is a Γ-morphism: $M_1 \to M_2$, when Γ is the set of positive formulas of Γ_0; in the uniqueness proof of Proposition III.3, we constructed an $L_{\mathcal{A}}$-morphism f: $M \to N$. The premise of the above statement (*) says that $\emptyset$ is a Γ-morphism: $M \to N$.

Lemma 2. *Let M, N be Σ-saturated models such that $\emptyset$ is a Γ-morphism: $M \to N$.*

(a) *Let a be an element of M such that whenever $\psi(v)\Gamma\varphi(v)$, there exist sentences $\rho, \rho' \in L_{\mathcal{A}}$ such that $M \models \psi(a) \to \rho\Gamma\rho'$ and $N \models \rho' \to \bigvee v\varphi$; then there exists $b \in N$ such that $\{(a, b)\}$ is a Γ-morphism: $M \to N$. Moreover,*

(i) *if* $\psi(v)\Gamma\varphi(v)$ *implies* $\bigvee v\psi\Gamma\bigvee v\varphi$, *then the condition on a is satisfied by any element of M*
(ii) *if* $\psi(v)\Gamma\varphi(v)$ *implies* $\psi(c)\Gamma\varphi(c)$ *and* $\psi(c')\Gamma\varphi(c')$, *then the condition on a is satisfied whenever* $M \models (a = c) \vee (a = c')$

(b) *Let b be an element of N such that whenever* $\psi(v)\Gamma\varphi(v)$, *there exists sentences* ρ, ρ' *such that* $M \models \bigwedge v\psi(v) \to \rho$, $\rho\Gamma\rho'$, $N \models \rho' \to \varphi(b)$. *Then there exists* $a \in M$ *such that* $\{(a, b)\}$ *is a* Γ*-morphism*: $M \to N$. *Moreover,*
(i) *if* $\psi\Gamma\varphi$ *implies* $\bigwedge v\psi\Gamma\bigwedge v\varphi$, *the condition on b is always satisfied; if* $\psi\Gamma\varphi$ *implies* $\bigwedge v(\theta \to \psi)\Gamma\bigwedge v(\theta \to \varphi)$, *the condition on b is satisfied whenever* $N \models \theta(b)$;
(ii) *if* $\psi\Gamma\varphi$ *implies* $\psi(c)\Gamma\varphi(c)$ *and* $\psi(c')\Gamma\varphi(c')$, *the condition on b is satisfied whenever* $N \models (b = c) \vee (b = c')$;
(iii) *if* $\psi\Gamma\varphi$ *implies* $\bigwedge v\psi\Gamma\bigvee v(F \wedge \varphi)$, *then there is an element b satisfying the condition, and such that* $N \models \bigvee vF(v) \to F(b)$.

Proof. (a) Let $p(v) = \{\theta \in L_{\mathscr{A}} : M \models \theta(a)\}$, and c be a constant not in L. By applying Lemma 1 to $\Psi = p(c)$ and Φ = the theory of N, we obtain sets $\Phi'(c)$ and $\Psi'(c)$ such that if a realizes $p(v) \cup \Psi'(v)$ in M, and b realizes $\Phi'(v)$ in N, then $\{(a, b)\}$ is a Γ-morphism: $M \to N$.

So it is enough to show that $\Psi'(v) \subset p(v)$ (hence a realizes $p(v) \cup \Psi'(v)$) and $\Phi'(v)$ is a Σ type in N (hence there is $b \in N$, realizing $\Phi'(v)$, because N is Σ-saturated). First $\Phi'(v)$ is a Σ set (because Γ is Σ and by Proposition III.2.a $p(v)$ is Σ); it is closed under $\bigwedge\!\!\!\bigwedge$ since Γ is, and if $\varphi \in \Phi'(v)$, by the assumption of (a) and the definition of $\Phi'(v)$, there are $\psi(v) \in p$ and sentences ρ, $\rho' \in L_{\mathscr{A}}$ such that $M \models \psi(a) \to \rho$ (hence $M \models \rho$), $\rho\Gamma\rho'$ (hence $N \models \rho'$) and $N \models \rho' \to \bigvee v\varphi(v)$. So $N \models \bigvee v\varphi$, which shows that Φ' is a Σ type in N.

Second, if $\neg\psi \in \Psi'(v)$, there exist (by the definition of Ψ') $\psi_0 \in p(v)$, φ_0 such that $\psi_0\Gamma\varphi_0$, and φ such that $\psi\Gamma\varphi$ and $N \models \bigwedge v(\varphi_0 \to \neg\varphi)$; $(\psi \wedge \psi_0)\Gamma(\varphi \wedge \varphi_0)$, hence by the assumption of (b) there are sentences ρ, ρ' such that $M \models \psi(a) \wedge \psi_0(a) \to \rho$ (hence $M \models \neg\rho \to \neg\psi(a)$), $\rho\Gamma\rho'$, and $N \models \rho' \to \bigvee v(\varphi \wedge \varphi_0)$; hence $N \models \neg\rho'$, and $\emptyset$ being a Γ-morphism $M \models \neg\rho$; so $M \models \neg\psi(a)$, and $\neg\psi(v) \in p$. So $\Psi' \subset p$.

The second part of (b) follows by remarking that we can satisfy the condition on a:

(i) taking $\rho = \bigvee v\psi$ and $\rho' = \bigvee v\varphi$
(ii) taking $\rho = \psi(c) \vee \psi(c')$ and $\rho' = \varphi(c) \vee \varphi(c')$

(b) The first part of (b) follows from (a) by duality (applying (a) when M is replaced by N, N by M, and Γ by the relation Γ^* defined by $\varphi\Gamma^*\psi \Leftrightarrow \neg\psi\Gamma\neg\varphi$).

The second part of (b) holds because in each case we satisfy the condition on b,

(i) by taking $\rho = \bigwedge v \psi$, $\rho' = \bigwedge v \varphi$, or $\rho = \bigwedge v(\theta \to \psi)$ and $\rho' = \bigwedge v(\theta \to \varphi)$

(ii) by taking $\rho = \psi(c) \wedge \psi(c')$, $\rho' = \varphi(c) \wedge \varphi(c')$

(iii) by taking for b any element which realizes the type $\{\bigvee v F(v) \to F(v_0)\}$ in N, and $\rho = \bigwedge v \psi$, $\rho' = \bigvee v(F \wedge \varphi)$.

Remark 3. We often apply Lemma II.C with M, N Σ-saturated; use of the following device will then ensure that its hypothesis (*) holds: *Suppose M Σ-saturated, X a subset of $|M|$ containing elements a, a' such that: $M \models a \neq a'$, and for all a_1, $M \models (a_1 = a) \vee (a_1 = a')$ implies $a_1 \in X$. Then any element of the boolean algebra $\mathbb{B}$ of M is of the form $(a_1 = a)_M$, with $a_1, a \in X$.*

Proof. Use the same argument as in the proof of Remark III.4.

(B) The "converse" to the proofs of characterization results by Σ-saturated models consists – for the case we consider in (A) – in proving (**) $\Rightarrow$ (*):

Proposition 4. *We say that the relation $\mathcal{R}(M, N)$ is projective* [8] *if there exists $L' \supset L$, a Σ theory T in $L'_{\mathscr{A}}$, and formulas $P_1(v)$, $P_2(v) \in L'_{\mathscr{A}}$ such that $\mathcal{R}(M, N) \Leftrightarrow$ there exists a model A of T such that $(A \restriction P_1) \restriction L$ is isomorphic to M, and $(A \restriction P_2) \restriction L$ to N.*

If $\mathcal{R}$ is projective, (**) $\Rightarrow$ (*).

Proof. Assume (**) and the premise of (*): $\emptyset$ is a Γ-morphism: $M \to N$. Let T_1, T_2 be the sets of sentences of $L_{\mathscr{A}}$ which hold respectively in M and in N, and consider any sentences ψ, φ of $L_{\mathscr{A}}$ such that

(i) either $\psi \in T_1$ and $\neg\varphi$ is consistent with T_2, or ψ is consistent with T_1 and $\neg\varphi \in T_2$.

Then ψ and φ do not satisfy the conclusion of (**). So ψ, φ satisfy the negation of the hypothesis of (**): there exists $M' \models \psi$, $N' \models \neg\varphi$ such that $\mathcal{R}(M', N')$. In other words, $T \cup \{\psi^{(P_1)}, \neg\varphi^{(P_2)}\}$ is consistent whenever ψ and φ satisfy (i). By compactness, $T \cup T_1^{(P_1)} \cup T_2^{(P_2)}$ is consistent and a formula $\theta^{(P_i)}$ ($\theta \in L_{\mathscr{A}}$, $i = 1$ or 2) is a consequence of this theory if and only if $T_i \vdash \theta$.

Let A be a Σ-saturated model realizing $T \cup T_1^{(P_1)} \cup T_2^{(P_2)}$, and let M' be $(A \restriction P_1) \restriction L$, N' be $(A \restriction P_2) \restriction L$. M' and N' are Σ-saturated by Proposition

[8] For brevity, we assume for this definition that constant symbols of L have been eliminated to the benefit of relation symbols, in the standard way.

III.2.c, and realize the theories T_1, T_2 respectively. By Proposition III.3 M' and M are isomorphic, and so are N' and N.

Examples. Well-known cases of (**) (proved essentially in [13, 14]) are, for sentences ψ, φ of $L_{\mathcal{A}}$:

φ is true in any homomorphic image of a model of ψ if and only if there exists a positive $\theta \in L_{\mathcal{A}}$ such that $\vdash (\psi \to \theta) \wedge (\theta \to \varphi)$.

φ is true in any extension of a model of ψ if and only if there exists $\theta \in \Sigma_1$ such that $\vdash (\psi \to \theta) \wedge (\theta \to \varphi)$.

In each of these two cases of (**), the relation $\mathcal{R}$ is projective, so to speak by definition, hence the corresponding cases of (*) hold:

Corollary 4. *Let M, N be Σ-saturated and denumerable; N is a (relative) homomorphic image of M if and only if for every positive $\theta \in L_{\mathcal{A}}$, $M \models \theta \Rightarrow N \models \theta$ and $N \models \neg\theta \Rightarrow M \models \neg\theta$. M is isomorphic to a submodel of N if and only if for every formula θ of Σ_1, $M \models \theta \Rightarrow N \models \theta$ and $N \models \neg\theta \Rightarrow M \models \neg\theta$.*

We give the first result for comparison with what we recalled from Keisler [6] in the introduction. From the second follows a result of Nebres [4]:

Proposition 5. *We say that N is a union of an n-family of models of ψ if for any $\bar{a} \in N$ there exists a model $B \models \psi$ such that $\bar{a} \in B \subset N$. The following are equivalent: every union of an n-family of models of ψ is itself a model of ψ; ψ is equivalent to a formula $\bigwedge \bar{x}\, \theta(\bar{x})$, where $\theta(\bar{x}) \in \Sigma_1$.*

Proof. Assume ψ is not equivalent to such a sentence $\bigwedge \bar{x}\theta$: then there is a model N satisfying $\neg\psi$ and all these sentences; we choose any sequence $\bar{c} \in N$. Without loss of generality we can suppose that $c_0, ..., c_{n-1}$ are constants of L which do not occur in ψ. We then apply Lemma 1, taking $\Gamma = \Sigma_1$, $\Psi = \{\psi\}$, $\Phi =$ the theory of N. Note that in this case the set Φ' defined in the Lemma is included in Φ: if $\theta \in \Sigma_1$ and $\psi \vdash \theta$, we can write $\theta = \theta'(\bar{c})$ where $\bar{c}$ does not occur in $\theta'(\bar{x})$; then $\psi \vdash \bigwedge \bar{x}\theta'(\bar{x})$, so $N \models \bigwedge \bar{x}\theta'(\bar{x})$ and $\theta'(\bar{c}) \in \Phi$. Thus, by the Lemma, if M is a model of theory $\{\psi\} \cup \Psi'$, then $\emptyset$ is a Σ_1-morphism: $M \to N$.

By Corollary 4, if we take M and N Σ-saturated, then M is isomorphic to a submodel B of N, which contains $\bar{c}$, as $c_0 \ldots c_{n-1}$ are constants of L. We have shown that N is n-union of models of ψ, although $N \models \neg\psi$, and this proves (the contrapositive of) the proposition.

Note that the proof actually yields a much more general result, as it still works if we replace everywhere the relation "M_1 is isomorphic to a submodel of M_2" by any projective relation $\mathcal{R}(M_1, M_2)$, and the set Σ_1 by any relation Γ (as in Definition 1), provided (**) holds for $\mathcal{R}$ and Γ.

We now give two applications of the procedure described in (A) (but we won't take care to state explicitly the two instances of (*) that are proved in order to obtain the two results). Another application shall be Lemma 11.

Local definability. Let L^0 be a sublanguage of L, and $\psi(v)$ a formula of $\mathrm{L}_{\mathscr{A}}$. ψ is locally definable in $\mathrm{L}^0_{\mathscr{A}}$ if for every model A there exists $\theta(v) \in \mathrm{L}^0_{\mathscr{A}}$ such that $A \models \bigwedge v(\psi \leftrightarrow \theta)$.

Proposition 6. *ψ is locally definable in $\mathrm{L}^0_{\mathscr{A}}$ if and only if ψ is preserved under (relative) L^0-automorphisms of boolean models.*

Proof. Assume ψ non definable in this sense: there exists a model A of theory $T = \{\neg \bigwedge v(\psi \leftrightarrow \theta) \colon \theta(v) \in \mathrm{L}^0_{\mathscr{A}}\}$. We set $\Theta_0 = \{\theta(v) \in \mathrm{L}^0_{\mathscr{A}} \colon T \cup \{\psi\} \vdash \theta\}$, $\Theta_1 = \{\theta(v) \in \mathrm{L}^0_{\mathscr{A}} \colon T \cup \Theta_0 \cup \{\neg\psi\} \vdash \theta\}$ One can check (in a way similar to Lemma 1): $\Theta_1 \cup \{\psi\}$ and $\Theta_1 \cup \{\neg\psi\}$ are Σ-types in M; for any $\theta(v) \in \mathrm{L}^0_{\mathscr{A}}$, $\Theta_1 \cup \{\psi\} \vdash \theta \Leftrightarrow \theta \in \Theta_1 \Leftrightarrow \Theta_1 \cup \cup \{\neg\psi\} \vdash \theta$.

Assume that A is Σ-saturated and denumerable. There exists a_0, $b_0 \in A$ of types $\{\psi\} \cup \Theta_1$ and $\{\neg\psi\} \cup \Theta_1$ respectively. We construct inductively $\{a_n ; n < \omega\}$ and $\{b_n ; n < \omega\}$, both enumerating $|A|$, such that the function $f_n \colon a_i \to b_i$ $(i < n)$ is an $\mathrm{L}^0_{\mathscr{A}}$-morphism: $A \to A$. f_1 is already constructed, assume f_n constructed, for an arbitrary value of $n \geq 1$.

If $n = 2k$, we take for a_n the k^{th} element of a fixed enumeration of $|A|$, and for b_n any element which makes f_{n+1} an $\mathrm{L}^0_{\mathscr{A}}$-morphism: $A \to A$. The existence of b_n follows from Lemma 2: add new constants $e_0 \ldots e_{n-1}$ to L, and let M, N be the models such that $M \upharpoonright \mathrm{L} = A = N \upharpoonright \mathrm{L}$, and $(e_i)_M = a_i$, $(e_i)_N = b_i$ $(i < n)$; then our hypothesis that f_n is an $\mathrm{L}^0_{\mathscr{A}}$-morphism: $A \to A$ is equivalent to saying that $\emptyset$ is a $\mathrm{L}^0_{\mathscr{A}}(\bar{e})$-morphism: $M \to N$. And Lemma 2(a)(i) applied with $\Gamma = \mathrm{L}^0_{\mathscr{A}}(\bar{e})$ gives that for any element a_n of M, there exists $b_n \in N$ such that $\{(a_n, b_n)\}$ is an $\mathrm{L}^0_{\mathscr{A}}(\bar{e})$-morphism: $M \to N$, which is equivalent to saying that f_{n+1} is an $\mathrm{L}^0_{\mathscr{A}}$-morphism: $A \to A$.

If $n = 2k + 1$, we take for b_n the k^{th} element of the fixed enumeration of $|A|$, and for a_n any element such that f_{n+1} then becomes an $\mathrm{L}^0_{\mathscr{A}}$-

morphism: $A \to A$. The existence of a_n follows from Lemma 2(b)(i) applied with $\Gamma = \mathrm{L}^0_{\mathscr{A}}(\bar{e})$ (interpreting $e_0 \ldots e_{n-1}$ in the same way).

Thus we obtain an $\mathrm{L}^0_{\mathscr{A}}$-morphism $f = \bigcup_n f_n$ from A onto A; by Lemma II.C applied with $M = N = A$ and $\Gamma = \mathrm{L}^0_{\mathscr{A}}$, f is a relative automorphism of $A \restriction \mathrm{L}^0$ (the hypothesis (*) of Lemma II.C holds by Remark 3). Since $A \models \psi(a_0) \wedge \neg\psi(f(a_0))$, the result follows in contrapositive form.

The relation $\psi \Rightarrow \varphi$. For sentences ψ, φ of $\mathrm{L}_{\mathscr{A}}$, we write $\psi \Rightarrow \varphi$ if for every ω_1-chain $\{B_\alpha ; \alpha < \omega_1\}$ of models of ψ, $\bigcup_{\alpha<\omega_1} B_\alpha$ is a model of φ (thus $\psi \Rightarrow \psi$ reads: ψ is preserved under unions of ω_1-chains).

$\{u_n ; n < \omega\}$ and $\{v_n ; n < \omega\}$ being two disjoint infinite sets of variables, we define a syntactic relation $\psi \Gamma \varphi$ on formulas of $\mathrm{L}_{\mathscr{A}}$ by the following inductive clauses:

(i) if ψ is quantifier free, $\psi \Gamma \psi$

(ii) if $\{(\psi_i, \varphi_i); i \in I\} \in \mathscr{A}$ and for all $i \in I$, $\psi_i \Gamma \varphi_i$, then $\bigwedge_I \psi_i \Gamma \bigwedge_I \varphi_i$ and $\bigvee_I \psi_i \Gamma \bigvee_I \varphi_i$

(iii) if $\psi(\overline{u v}) \Gamma \varphi(\overline{u v})$ and $F(\bar{u}) \in \mathrm{L}_{\mathscr{A}}$, $i < n$, then $\bigwedge u_i \psi(\bar{u}, \bar{v}) \Gamma \bigvee u_i [F'(\bar{u}) \wedge \varphi(\overline{u v})]$, where F' is $(\bigvee u_i F) \to F$;

(iv) if $\psi(\overline{u v}) \Gamma \varphi(\overline{u v})$, then for $i < n$, $\bigvee v_i \psi(\overline{u v}) \Gamma \bigvee v_i \varphi(\overline{u v})$.

Proposition 7. *For $\psi, \varphi \in \mathrm{L}_{\mathscr{A}}$, the following are equivalent* (a) $\psi \Rightarrow \varphi$, (b) *there exist ψ', φ' such that $\psi \vdash \varphi'$, $\psi' \Gamma \varphi'$ and $\varphi' \vdash \varphi$.*

Proof. In a way quite similar to Proposition IV.14, we have: Λ *$\psi \Rightarrow \varphi$ if and only if for every sandwich (A_0, B, A_1), $B \models \psi$ implies* $A_1 \models \varphi$. Then the result follows immediately from the

Lemma 7. *For ψ, φ sentences of $\mathrm{L}_{\mathscr{A}}$, the following are equivalent:*

(a) *For any sandwich (A_0, B, A_1), $B \models \psi \Rightarrow A_1 \models \varphi$*

(b) *there exists ψ', φ' such that $\psi \vdash \psi'$, $\psi' \Gamma \varphi'$ and $\varphi' \vdash \varphi$.*

Proof. (b) $\Rightarrow$ (a). A straightforward induction on the clauses defining Γ shows that if $\psi(\overline{u v}) \Gamma \varphi(\overline{u v})$ and $\bar{a} \in A_0$, $\bar{b} \in B$, then $B \models \psi(\overline{a b}) \Rightarrow$ $\Rightarrow A_1 \models \varphi(\overline{a b})$. Hence (b) $\Rightarrow$ (a).

$\neg$(b) $\Rightarrow$ $\neg$(a). Assume ψ and φ satisfy $\neg$(a), and let Ψ', Φ' be the sets obtained by applying Lemma 1 with $\Psi = \{\psi\}$ and $\Phi = \{\neg\varphi\}$: if B, A_1 are models of theories $\{\psi\} \cup \Psi'$ and $\{\neg\varphi\} \cup \Phi'$ respectively, then $\emptyset$ is a Γ-morphism: $B \to A_1$.

We assume B and A_1 Σ-saturated and construct by induction on n two sequences $\{a_n ; n < \omega\}$, $\{b_n ; n < \omega\}$ in B and two sequences $\{a'_n ; n < \omega\}$, $\{b'_n ; n < \omega\}$ in A_1, such that:

whenever $\psi'(\overline{uv})\Gamma\varphi'(\overline{uv})$, *then* $B \models \psi'(\overline{a}\,\overline{b}) \Rightarrow A_1 \models \varphi'(\overline{a'}\,\overline{b'})$, *and* $A_1 \models \neg\varphi'(\overline{a'}\,\overline{b'}) \Rightarrow B \models \neg\,\psi'(\overline{a}\,\overline{b})$.

(In other words, the correspondence which sends a_n to a'_n and b_n to b'_n is a Γ-morphism: $M \to N$, in a slightly extended sense). The induction hypothesis on n is that for $i < n$, the elements a_i, a'_i, b_i, b'_i are chosen with this property. It holds if $n = 0$, as $\emptyset$ is a Γ-morphism: $B \to A_1$; we assume it for an arbitrary value of n, and assume that an ω-enumeration of $|B|$ has been fixed, as well as an enumeration $\{F_n(\overline{u}, x); n < \omega\}$ of all formulas of $L_{\mathscr{A}}$ with free variables as indicated.

Choice of a_n, a'_n. By enriching L and the models B, A_1 with new constants, we may assume without loss of generality that for all $i < n$ there are constants c_i, d_i of L such that $c_{iB} = a_i$, $d_{iB} = b_i$ and $c_{iA_1} = a'_i$, $d_{iA_1} = b'_i$; then the induction hypothesis becomes equivalent to "$\emptyset$ is a Γ-morphism: $B \to A_1$", and Lemma 2(b)(iii) applied with $M = B$, $N = A_1$, and $F = \mathbf{V}x\,F_n \to F_n(\overline{u}\,u_n)$ allows to choose a_n, a'_n so that $A_1 \models \mathbf{V}x\,F_n(\overline{a'}x) \to F_n(\overline{a'}a'_n)$ and that the induction hypothesis holds for a_i $(i \leq n)$ and b_i $(i < n)$.

N.B. Whenever F_n is equivalent to $(x = u_i) \vee (x = u_j)$, we can take for a'_n any element a such that $A_1 \models (a = a'_i) \vee (a = a'_j)$: because we can then apply 2(b)(ii) instead of 2(b)(iii) to obtain the corresponding element a_n.

Choice of b_n, b'_n. b_n is the first element of $|B|$ (in the fixed enumeration of $|B|$), which is not in $\{a_i : i \leq n\} \cup \{b_i ; i < n\}$. Then b'_n can be chosen in a way that the induction hypothesis holds for $n + 1$: this follows from Lemma 2(b)(i) (enriching L and the models in the way we did when we chose a_n, a'_n).

So the construction can be done; moreover, we can use the freedom indicated by the above Nota Bene to the effect that for all $i < j < \omega$, $A_1 \models (a = a'_i) \vee (a = a'_j)$ implies $a \in \{a_n ; n < \omega\}$.

Let A_0 be $A_1 \restriction \{a_n ; n < \omega\}$; by construction A_0 is an $L_{\mathscr{A}}$-submodel of A_1. Let f be the function such that $f(a_n) = a'_n$ and $f(b_n) = b'_n$ for all $n < \omega$; by construction f is a function of B onto a subset of A_1 containing $|A_0|$, and is a Γ-morphism: $B \to A_1$. By clause (i) of the definition of Γ, this implies that f is a Σ_0-morphism: $B \to A_1$, Σ_0 being the set of quantifier free formulas of $L_{\mathscr{A}}$. Lemma II.C applied with $M = B$, $N = A_1$ and $\Gamma = \Sigma_0$ shows that f is a relative isomorphism of B onto a submodel B' of A_1 (the hypothesis (*) of the lemma holds by Remark 3 applied to B with $X = |B|$ and to A_1 with $X = |A_0|$). So (A_0, B', A_1) is a sandwich such that $B' \models \psi$, yet $A_1 \models \neg\varphi$; which shows $\neg(b) \Rightarrow \neg(a)$.

Interpolation results. We first indicate the pattern of proof that we shall follow here. We consider a relation $\mathcal{R}(M, N)$, on models of the same language, and a syntactically defined subset Δ of $L_{\mathscr{A}}$; and we prove

Characterization lemma. (a) *Any sentence $\theta \in \Delta$ is preserved under $\mathcal{R}$ (that is, $M \models \theta$ and $\mathcal{R}(M,N)$ implies $N \models \theta$).*

(b) *Consider ψ, $\varphi \in L_{\mathscr{A}}$ and L^0, a sublanguage of L; if $M \models \psi$ and $\mathcal{R}(M \restriction L^0, N \restriction L^0)$ always implies $N \models \psi$, then there exists $\theta \in \Delta \cap L^0_{\mathscr{A}}$ such that $\vdash (\psi \to \theta) \wedge (\theta \to \varphi)$.*

This lemma, when $L^0 = L$, is a case of the result (**) considered in (A) and (B), but when $L^0 \not\subset L$ it also gives an interpolation result, provided one has the:

"Extension lemma" – Suppose $\mathcal{R}(A \restriction L^0, B_0)$, where B_0 is a model of L^0 and A a model of a language containing L^0; then there exists B_1 such that $B_1 \restriction L^0 = B_0$ and $\mathcal{R}(A, B_1)$.

Indeed, whenever these two lemmas hold, one has the:

Interpolation theorem. *For any formula ψ, let $L(\psi)$ be the language whose symbols are = plus the relation and constant symbols that appear in ψ. A formula θ is an interpolant for ψ^1, ψ^2 if $L(\theta)$ is a sublanguage of $L(\psi^1)$ and $L(\psi^2)$, and $\vdash (\psi^1 \to \theta) \wedge (\theta \to \psi^2)$. Then if $\vdash \psi^1 \to \psi^2$ and $\psi^1 \in \Delta$, then ψ^1 and ψ^2 have an interpolant in Δ.*

Proof. Assume $\psi^1 \in \Delta$, and ψ^1, ψ^2 have no interpolant in Δ; by the Characterization Lemma (b), applied with $\psi = \psi^1$ $\varphi = \psi^2$ and $L^0 = L(\psi^1) \cap L(\psi^2)$, there exists $M \models \psi^1$ and $N \models \neg\psi^2$ such that $\mathcal{R}(M \restriction L^0, N \restriction L^0)$. By the Extension lemma when $A = M \restriction L(\psi^1)$ and $B_0 = N \restriction L^0$, there exists a model B_1 such that:

–$\mathcal{R}(M \restriction L(\psi^1), B_1)$; hence by the Characterization Lemma (a), $M \models \psi^1$ implies $B_1 \models \psi^1$;

– and $B_1 \restriction L^0 = N \restriction L^0$; hence B_1 can be enriched to a model of L satisfying (as N) $\neg\psi^2$.

Thus $B_1 \models \psi^1 \wedge \neg\psi^2$ and, $\psi^1 \to \psi^2$ being non valid, the theorem is proved in contrapositive form.

In [2], Makkai considers relations $\mathcal{R}_i$ $(i \leq 8)$ and corresponding sets of formulas Δ_i, and proves the above "Characterization Lemma", when $\mathcal{R}=\mathcal{R}_i$, $\Delta = \Delta_i$ and $L^0 = L$. But a simple inspection of the proofs shows that they work without change if $L^0 \neq L$; so we shall admit that this Lemma holds when $\mathcal{R} = \mathcal{R}_i$, $\Delta = \Delta_i$ $(i \leq 8)$. In addition, the "Extension Lemma" is easy to check in the cases $i = 5$ and $i = 7$; hence we have the following instances of the above "Interpolation theorem".

Proposition 8. *The "Interpolation theorem" holds in the following cases:* $\Delta = \Delta_5$, $\Delta = \Delta_7$, *where* Δ_i *is defined in Makkai* [2, p. 312].

We next use the notation H_Γ as defined on p. 51, taking $\Gamma_0 = L_{\mathcal{A}}$. And we denote by H_L the set H_Γ when Γ is the set of atomic formulas of L.

Lemma 9. *Let* Γ *be a* Σ *subset of* $L_{\mathcal{A}}$, *containing the atomic formulas.*

(a) *If* M *is an* $L_{\mathcal{A}}$*-model and* M/D *a* Γ*-quotient of* M *then for any* $\theta \in H_\Gamma$, $M \models \theta \Rightarrow M/D \models \theta$.

(b) *Conversely, if* $\psi, \varphi \in L_{\mathcal{A}}$, *and* φ *is true in any* Γ*-quotient of an* $L_{\mathcal{A}}$*-model of* ψ, *then there is* $\theta \in H_\Gamma$ *such that* $\vdash (\psi \to \theta) \wedge (\theta \to \varphi)$.

Proof. (a) is the same as Lemma II.B, (i) $\Rightarrow$ (ii).
(b) Assume that there is no θ in H_Γ such that $\vdash (\psi \to \theta) \wedge (\theta \to \varphi)$; by compactness, if Θ is the set of sentences of H_Γ which are consequences of ψ, then $\{\neg\varphi\} \cup \Theta$ is a theory; we let M and N be Σ-saturated models realizing the theories $\{\psi\}$ and $\{\neg\varphi\} \cup \Theta$ respectively. By induction on n, we construct enumerations $\{a_n ; n < \omega\}$ of $|M|$, $\{b_n ; n < \omega\}$ of $|N|$ such that the function f: $a_n \to b_n$ $(n < \omega)$ shall satisfy:
(*) for all $\theta \in H_\Gamma(\mathrm{dom} f)$, $M \models \theta \Rightarrow N \models f\theta$;
the induction hypothesis is: (*) *holds for* $f \restriction \bar{a}$. By the choice of the theories of M and N, it is true for $n = 0$; we assume it for an arbitrary n.

If $n = 2k$, we take for a_n the $(1 + k)^{\text{th}}$ element of a fixed enumeration of $|M|$; we let $p(v_0)$ be $\{\theta(v_0) \in H_\Gamma(\bar{a}) : M \models \theta(a_n)\}$. From the induction hypothesis it follows that for any θ in p, $N \models \mathbf{V}v_0 f\theta$; hence fp is a Σ type in N and we take for b_n an element realizing fp in N: thus the induction hypothesis becomes true for $n + 1$.

If $n = 2k + 1$, we take for b_n the $(1 + k)^{\text{th}}$ element of a fixed enumeration of $|N|$; we take for a_n an element of M realizing the type $\{\mathbf{T}\}$ over $\bar{a}$, that is: for any $\theta(v_0) \in H_\Gamma(\bar{a})$, $M \models \theta(a_n)$ if and only if $M \models \bigwedge v_0\,(\mathbf{T} \to \theta)$, that is $M \models \bigwedge v_0\,\theta(v_0)$. So if $\theta \in H_\Gamma(\bar{a})$, $M \models \theta(a_n)$ implies $M \models \bigwedge v_0\,\theta(v_0)$, which by the induction hypothesis implies $N \models f \bigwedge v_0\,\theta$, hence $N \models f\theta(b_n)$: the induction hypothesis becomes true for $n + 1$.

Thus we obtain f, from M onto N, satisfying (*); applying the Rasiowa–Sikorski theorem (II.3), we take an $\mathrm{L}_{\mathscr{A}}$-quotient N/D_1 of N. Then (*) remains true when N/D_1 replaces N. So by Lemma II.B.b there exists a Γ-quotient M/D of M such that f becomes an isomorphism between M/D and N/D_1. Thus $M \models \psi$ and $M/D \models \neg\varphi$, which shows (b) in contrapositive form.

Proposition 10. *The above "Interpolation theorem" is true when* $\Delta = H_{\mathrm{L}}$ *and* $\Delta = H_{\mathrm{L}_\omega}$.

Proof. Lemma 9 implies the two following cases of the above "Characterization Lemma":

$\Delta = H_{\mathrm{L}}$, and $\mathcal{R}(M, N) \Leftrightarrow$ "M is an $\mathrm{L}_{\mathscr{A}}$-model and there is a filter D such that $N = M/D$";

$\Delta = H_{\mathrm{L}_\omega}$, and $\mathcal{R}(M, N) \Leftrightarrow$ "M is an $\mathrm{L}_{\mathscr{A}}$-model and there is an ultrafilter D such that $N = M/D$".

And in these two cases, the above "Extension Lemma" is easily checked.

The "Interpolation theorem" follows.

Many-sorted languages. A *many-sorted language* L is a language in the usual sense, together with a partition of the set of variables into infinite sets called *the sorts of* L, and a function which associates one sort to each constant of L. The notion of a $\mathbb{B}$-*valued model* M of L is the same as before, except for the following points: to each sort i of L, M assigns a set M^i, and $|M|$ denotes $\bigcup\{M^i : i \text{ sort of L}\}$; and for each constant c of sort i, c_M must be in M^i. The *interpretation of formulas in M* is the same as before, except that if v is a variable of sort i, $\bigwedge v\,\psi(v)_M$ is $\bigcap\{\psi(a)_M : a \in M^i\}$.

For M model of L, we define "*M is Σ-saturated*" by adding only to the previous definition that $M^i \cap M^j$ is empty for distinct sorts i, j. This seems too restrictive since we require it only for Σ-saturated models, but is not since M is not supposed to be an equality model: instead of one element $a \in M^i \cap M^j$, we use two distinct elements a^i, a^j, but set $M \models a^i = a^j$. *Then the previous results on Σ-saturated models extend* without any change.

Let L be a many sorted language of $\mathcal{A}$ (without functions). We shall follow, but not literally, the pattern described on p. 83, to arrive at the interpolation theorem for L: the first step is to define our relation $\mathcal{R}$.

Notations. (a) A formula is called *basic* if it is atomic or the negation of an atomic formula.

(b) We consider a set C of basic formulas; M, N being $\mathbf{2}$-valued models of L and f a function such that $\mathrm{dom} f \subset |M|$, $\mathrm{im} f \subset |N|$, and $f(a) \in N^i$ if $a \in M^i$, we say that *f is a C-homomorphism between M and N* if for any constant c which occurs in C, $c_M \in \mathrm{dom} f$ and $f(c_M) = c_N$, and if for every sentence $\varphi \in C\,(\mathrm{dom} f)$, $M \models \varphi \Rightarrow N \models f\varphi$.

(c) We consider two sets I, I' of sorts of L; the relation $\mathcal{R}\,(M, N)$ is the conjunction of the three relations:

(1) $1_{|M| \cap |N|}$ is a C-homomorphism between M, N;

(2) for each $i \in I$, $M^i \subset N^i$; we shall denote such a property by $M^I \subset N^I$,

(3) $M^{I'} \supset N^{I'}$.

(d) Then the set Δ which is associated (as in p. 83) to $\mathcal{R}$ is the closure of $C \cup \{\perp\}$ in $\mathrm{L}_{\mathcal{A}}$, under $\bigwedge\!\!\bigwedge$, $\bigvee\!\!\bigvee$, $\bigvee u$ (where the sort of u belongs to I) and $\bigwedge u$ (where the sort of u belongs to I').

Characterization lemma 11. *We assume that $\varphi(t) \in C$ implies $\varphi(s) \in C$ whenever s is of the same sort as t and s is a constant occurring in C or a variable; we assume also that if the equality sign occurs in C then the formulas $(u = v)$, $\neg(u = v)$ belong to C, for all variables u, v.*

(a) *Any sentence $\theta \in \Delta$ is preserved under $\mathcal{R}$*

(b) *If $\psi, \varphi \in \mathrm{L}_{\mathcal{A}}$, if $M \models \psi$ and $\mathcal{R}(M, N)$ imply $N \models \varphi$, then there exists $\theta \in \Delta$ such that $\vdash (\psi \to \theta) \wedge (\theta \to \varphi)$.*

Proof. (a) Straightforward.

(b) We add to L a new sort, $*$, and two constants of sort $*$, denoted 0^* and 1^*. We add to C all basic sentences that use only $=$ and terms of sort $*$, we define Δ' as we did for Δ, but from the extended set C, and we denote by F the following formula:

$0^* \neq 1^* \wedge \bigwedge v^*(v^* = 1^* \vee v^* = 0^*)$. If θ is a sentence of Δ', by substituting $\perp$ to every occurrence of the atomic sentence $0^* = 1^*$ in θ, we obtain a formula $\theta_1 \in \Delta$ such that $F \vdash \theta \leftrightarrow \theta_1$.

Now we assume that for every $\theta \in \Delta$, $(\psi \to \theta) \wedge (\theta \to \varphi)$ is non valid; from what precedes follows that for any $\theta \in \Delta'$, $(\psi \wedge F \to \theta) \wedge (\theta \wedge F \to \varphi)$ is non valid. Then by Lemma 1 applied with $\Gamma = \Delta'$, $\Psi = \{\psi, F\}$, $\Phi = \{\neg\varphi, F\}$, there exists models N, realizing the theory $\{\neg\varphi, F\} \cup \cup \{\theta \in \Delta' : \psi \wedge F \vdash \theta\}$, and M, realizing the theory $\{\psi, F\} \cup \{\neg\theta : \theta \in \Delta', N \models \neg\theta\}$; moreover then $\emptyset$ is a Δ'-morphism: $M \to N$. We assume in addition that M, N are Σ-saturated and that for any constants $c \neq c' \in \mathrm{L}$, $c_M \neq c'_M$ and $c_N \neq c'_N$ (this can be done because of Condition III.3.ii). Then the function $f_0 : c_M \to c_N$ (c occuring in C) is injective and is a Δ'-morphism: $M \to N$ since $\emptyset$ is one.

By induction on n, we construct $\{a_n ; n < \omega\} \subset |M|$ and $\{b_n ; n < \omega\} \subset |N|$ such that $f_n = f_0 \cup \{(a_i, b_i); i < n\}$ is an injective Δ'-morphism: $M \to N$. We assume this done for an arbitrary value of n.

If $n = 4k$, we choose $i \in I$ and take for a_n any element of M^i; then we can choose b_n in a way that f_{n+1} is a Δ'-morphism: $M \to N$; indeed, this follows from Lemma 2(a)(i), applied with $\Gamma = \Delta'(\bar{c})$ (enriching M and N by setting $c_{i_M} = a_i$, $c_{i_N} = b_i$); moreover b_n can be chosen so that f_{n+1} remains injective, by Condition III.3.ii.

If $n = 4k + 1$, we take for a_n any element of M^*; $b_n \in N^*$ can then be chosen so that f_{n+1} still satisfies the hypothesis, because of Condition 3.ii and of Lemma 2(a)(ii) applied with $\Gamma = \Delta'(\bar{c})$ (noting that $M \models (a_n = 1^*) \vee (a_n = 0^*)$).

If $n = 4k + 2$, we choose $i \in I'$ and take for b_n any element of N^i; then we can choose $a_n \in M^i$ so that f_{n+1} satisfies the induction hypothesis, because of Condition III.3.ii and of Lemma 2(b)(i), applied with $\Gamma = \Delta'(\bar{c})$.

If $n = 4k + 3$, we take for b_n any element of N^*; $a_n \in M^*$ can then be chosen so that f_{n+1} still satisfies the hypothesis, because of Lemma 2(b)(ii) applied with $\Gamma = \Delta'(\bar{c})$, noting that $N \models (b_n = 1^*) \vee (b_n = 0^*)$. Thus $f = \bigcup_n f_n$ can be constructed, and it is clear that the degrees of freedom in its construction can be used to the effect that:

(1) if $i \in I \cup \{*\}$, $M^i \subset \operatorname{dom} f$ and if $i \in I' \cup \{*\}$, $N^i \subset \operatorname{im} f$.

Let $\mathbb{B}$, $\mathbb{B}_1$ be the boolean algebras of M and N; we define $q : \mathbb{B} \to \mathbb{B}_1$ by

$$a, a' \in M^* \Rightarrow q[(a = a')_M] = (f(a) = f(a'))_N .$$

q is an isomorphism from $\mathbb{B}$ onto $\mathbb{B}_1$, by Lemma II.C. a applied here with $f \upharpoonright M^*$ instead of f, and with Γ equal to the set of formulas of Δ' which contain only the sort $*$. (The $(*)$ hypothesis of this Lemma holds by Remark 3 applied to M with $X = M^*$ and to N with $X = N^*$; the other because f is a Δ'-morphism: $M \to N$).

Moreover, if θ is a sentence of Δ' (dom$f$), by Remark 3 applied with $X = M^*$ there exists $a \in M^*$ such that $\theta_M = (a = 1^*)_M$. $M \models \neg(a = 1^*) \vee \theta$ and this formula belongs to $\Delta'(M)$, and f is a Δ'-morphism, so $N \models \neg(f(a) = 1^*) \vee f\theta$, in other words $(f(a) = 1^*)_N \subset (f\theta)_N$. Since by definition of q, $(f(a) = 1^*)_N = q(\theta_M)$, we have shown:
(2) for all $\theta \in \Delta'(\mathrm{dom} f)$, $q(\theta_M) \subset (f\theta)_N$.

We are now in a position to conclude: f being injective, we may assume that $f = 1_{|M| \cap |N|}$; then (1) becomes: $M^I \subset N^I$ and $M^{I'} \supset N^{I'}$. We may also assume that $\mathbb{B} = \mathbb{B}_1$, $q = 1_{\mathbb{B}}$; then (2) implies: [for all $\theta \in C'$ $(|M| \cap |N|)$, $\theta_M \subset \theta_N$], so that $1_{|M| \cap |N|}$ becomes a C-homomorphism between M and N. Thus $M \models \psi$, $N \models \neg\varphi$ and (for the extension of $\mathcal{R}$ to boolean models) $\mathcal{R}(M, N)$, which proves (b) in contrapositive form.

Interpolation theorem 12. *Assume* L *contains for each sort i at least one constant of sort i, and let ψ^1, ψ^2 be sentences of* $\mathrm{L}_{\mathcal{A}}$. *If $\vdash \psi^1 \to \psi^2$, there exists a sentence $\theta \in \mathrm{L}_{\mathcal{A}}$ such that $\vdash (\psi^1 \to \theta) \wedge (\theta \to \psi^2)$, and θ satisfies the interpolation clauses (a_ϵ), (b_ϵ), (c_ϵ), for $\epsilon = 1$ and $\epsilon = 2$:*

(a_ϵ) if $=$ occurs in θ then it occurs in ψ^1 or in ψ^1; for every other relation symbol R, *if* $\mathrm{R}(\bar{t})$ *occurs positively (resp. negatively) in θ, then there exists for each $i < n$ terms s_i of the same sort as t_i, such that* $\mathrm{R}(\bar{s})$ *occurs positively (resp. negatively) in ψ^ϵ*

(b_ϵ) for any sort i, if $\bigwedge^i$ occurs in θ then it occurs in ψ^ϵ and if $\bigvee^i$ occurs in θ, it occurs in ψ^ϵ

(c_1) for each sort i except if $\bigwedge^i$ occurs in ψ^1 but not in ψ^2, every constant of sort i which occurs in θ occurs in ψ^1

(c_2) for each sort i except if $\bigvee^i$ occurs in ψ^2 but not in ψ^1, every constant of sort i which occurs in θ occurs in ψ^2.

Proof. We define, for $\epsilon = 1, 2$,

C_ϵ = the set of basic sentences θ which satisfy the clauses (a_ϵ) and (c_ϵ)

I_ϵ = the set of sorts i such that $\bigvee^i$ occurs in ψ^ϵ

I'_ϵ = the set of sorts i such that $\bigwedge^i$ occurs in ψ^ϵ.

We are going to apply Lemma 11.b in the case:

$C = C_1 \cap C_2$, $I = I_1 \cap I_2$, $I' = I'_1 \cap I'_2$;

note that in this case the set Δ is exactly the set of formulas which satisfy all the interpolation conditions of the theorem. So if we assume that ψ^1, ψ^2 do not satisfy the conclusion of the theorem, there is no formula θ in Δ such that $\vdash (\psi^1 \to \theta) \wedge (\theta \to \psi^2)$, and Lemma 11(b) applied with $\psi = \psi^1$ and $\varphi = \psi^2$ in this case, shows that there exists models

$M \models \psi^1, N \models \neg \psi^2$, such that $\mathcal{R}(M, N)$, that is:

(1) $1_{|M| \cap |N|}$ is a $(C_1 \cap C_2)$-homomorphism between M and N

(2) $M^{I_1 \cap I_2} \subset N^{I_1 \cap I_2}, M^{I'_1 \cap I'_2} \supset N^{I'_1 \cap I'_2}$.

Now we want to construct a model B such that $\mathcal{R}(M, B)$ holds in the case $C = C_1, I = I_1, I' = I'_1$: then Lemma 11 (a) applied in this case with $\theta = \psi^1$, shows that $B \models \psi^1$. Moreover, $\mathcal{R}(B, N)$ holds in the case $C = C_2$, $I = I_2, I' = I'_2$: then Lemma 11 (a) applied in this case and with $\theta = \psi^2$ shows that $B \models \neg \psi^2$. This will show that $\psi^1 \to \psi^2$ is non valid, and will prove the theorem in contrapositive form.

As a step in the construction of B, we construct a model A of the restriction of L to the symbols that occur in $C_1 \cup C_2$, such that

(3) $1_{|M|}$ is a C_1-homomorphism between M and A, and $1_{|N|}$ is a C_2-homomorphism between A and N.

Construction of A. For each sort i, we set $A^i = M^i \cup N^i$. Let F denote the set of atomic formulas of L whith no occurrence of a constant symbol or of the equality sign. Suppose that f is any map from the atomic sentences of $F(A)$ into $\{0, 1\}$, and g is any map which for every sort i sends the constants of sort i of L into A^i. Then the following procedure defines a model A of L, which satisfies the equality axioms:

we set $\theta_A = f(\theta)$ for every $\theta \in \mathrm{dom} f$; and $c_A = g(c)$ for every constant c of L; and

(4) for every atomic sentence θ not in $F(A)$, we define θ_A in the unique way which satisfies the condition $\theta_A = \theta'(c_A)_A$, whenever $\theta = \theta'(c)$ and $c \in$ L.

So it remains to fix f and g for our purpose:

we set $g(c) = c_M$ if c occurs in C_1, and $g(c) = c_N$ otherwise; if θ is a sentence of $F(A)$ then we require

(5) $\theta \in C_1(M), M \models \theta \Rightarrow f(\theta) = 1; \neg\theta \in C_1(M), M \models \neg\theta \Rightarrow f(\theta) = 0$
$\neg\theta \in C_2(N), N \models \theta \Rightarrow f(\theta) = 1; \theta \in C_2(N), N \models \neg\theta \Rightarrow f(\theta) = 0$; in all other cases, we set (conventionally) $f(\theta) = 1$.

Note that (5) effectively defines a *function* f: indeed, if for a given sentence θ both $f(\theta) = 1$ and $f(\theta) = 0$ were required by (5), then either when $\epsilon\theta = \theta$ or when $\epsilon\theta = \neg\theta$, we would have:

$\epsilon\theta \in C_1(M) \cap C_2(N) = C(|M| \cap |N|)$, and $M \models \epsilon\theta, N \models \neg\epsilon\theta$; hence $1_{|M| \cap |N|}$ would not be a C-homomorphism, contrary to our assumption.

It is easy to see, for the model A we thus defined, that $1_{|M|}$ is a C_1-homomorphism between M and A: indeed $M \models \theta \Rightarrow A \models \theta$ is required by (5) for every sentence θ of $C_1(M)$ in which only elements of A occur as constants; and by (4), for the other sentences θ of $C_1(M)$. Similarly, $1_{|N|}$ is a C_2-homomorphism between A and N.

Construction of $|B|$. For each sort i we chose a set $B^i \subset M^i \cup N^i = A^i$, as indicated by the table below; and we set $|B| = \bigcup_i B^i$, $B = A \upharpoonright |B|$. We leave to the reader to check that

(6) B^i is non empty, and for each constant c of sort i occuring in $C_1 \cup C_2$, $c_A \in B^i$

(7) $M^{I_1} \subset B^{I_1}, M^{I'_1} \supset B^{I'_1}, B^{I_2} \subset N^{I_2}, B^{I'_2} \supset N^{I'_2}$.

Properties (3) and (6) imply that $1_{|M| \cap |B|}$ is a C_1-homomorphism between M and B, and that $1_{|B| \cap |N|}$ is a C_2-homomorphism between B and N; together with (7), these properties are equivalent to the relation: "$\mathcal{R}(M, B)$ holds when $C = C_2$, $I = I_2$, $I' = I'_2$ and $\mathcal{R}(B, N)$ holds when $C = C_2$, $I = I_2$, $I' = I'_2$". Thus B has the required properties.

The table indicates B^i as a function of the occurrence of $\bigvee^i$, $\bigwedge^i$ in ψ^1 and ψ^2; in its last column, remarks are given, which follow from (2) and allow us to show (6), (7).

$i \in I_1$	$i \in I'_1$	$i \in I_2$	$i \in I'_2$	B^i	Remarks
+	+	+	+	M^i	$M^i = N^i$
–	+	+	+	N^i	$N^i \subset M^i$
+	–	+	+	N^i	$N^i \supset M^i$
+	+	–	+	M^i	$M^i \supset N^i$
+	+	+	–	M^i	$M^i \subset N^i$
–	+	–	+	M^i	$M^i \supset N^i$
+	–	+	–	N^i	$N^i \supset M^i$
–	+	+	–	$M^i \cap N^i$	
+	–	–	+	$M^i \cup N^i$	
–	–	+	+	N^i	
+	+	–	–	M^i	
+	–	–	–	$M^i \cup N^i$	
–	+	–	–	M^i	
–	–	+	–	N^i	
–	–	–	+	$M^i \cup N^i$	
–	–	–	–	$M^i \cup N^i$	

N.B. Feferman's statement of the interpolation theorem for L, [1, p.32], follows easily from the present statement.

Remarks 13. (a) In [7], Keisler introduces the class of "conjunctive formulas", obtained from L_ω using arbitrary conjunctions and infinite well-ordered strings of quantifiers. And he proves some compactness and interpolation properties for these formulas, using saturated models. This can be imitated with Σ-saturated models, for a class **C** of formulas obtained this time from $L_{\mathscr{A}}$, using conjunctions and one ω-sequence of quantifiers.

(b) For Σ-saturated models,

if $\{\mathbf{V}v\theta;\theta \in p\}$ is satisfied, so is $\mathbf{V}v\bigwedge p$ – where p is a Σ subset of $L_{\mathscr{A}}$, closed under $\bigwedge$. Hence the adjunction of formulas of **C** and of such "deduction rules" to an ordinary deduction system for $L_{\mathscr{A}}$ yields a conservative extension of this system, although the added rules are not valid. The author – in his ignorance – wonders if it would be of some interest to study deductive systems enlarged in such a way.

Added in proof (Sept. 19, 1973). Theorems 7 and 8 of §IV (with somewhat different theories T_0, T_1) are announced by J. Gregory in Notices of the Ann. Math. Soc. 17, pp. 967–968.

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